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Hydrodynamics of the cascading plasma

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Abstract

The cascading gauge theory of Klebanov *et.al* realizes a soluble example of gauge/string correspondence in a non-conformal setting. Such a gauge theory has a strong coupling scale Λ, below which it confines with a chiral symmetry breaking. A holographic description of a strongly coupled cascading gauge theory plasma is represented by a black brane solution of type IIB supergravity on a conifold with fluxes. A characteristic parameter controlling the high temperature expansion of such plasma is $\mathcal{Q} \simeq \left(\ln \frac{T}{\Lambda}\right)^{-1}$. In this paper we study the speed of sound and the bulk viscosity of the cascading gauge theory plasma to order $\mathcal{O}(\mathcal{Q}^4)$. We find that the bulk viscosity satisfies the bound conjectured in arXiv:0708.3459. We comment on difficulties of computing the transport coefficients to all orders in $\mathcal{Q}$. Previously, it was shown that a cascading gauge theory plasma undergoes a first-order deconfinement transition with unbroken chiral symmetry at $T_{critical} = 0.6141111(3)\Lambda$. We show here that a deconfined chirally symmetric phase becomes perturbatively unstable at $T_{unstable} = 0.8749(0)T_{critical}$. Near the unstable point the specific heat diverges as $c_V \sim |1 - \frac{T_{unstable}}{T}|^{-1/2}$.

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1 Introduction

Consider $\mathcal{N}=1$ four-dimensional supersymmetric $SU(K+P)\times SU(K)$ gauge theory with two chiral superfields A_1, A_2 in the $(K+P, \overline{K})$ representation, and two fields B_1, B_2 in the $(\overline{K+P}, K)$. This gauge theory has two gauge couplings g_1, g_2 associated with two gauge group factors, and a quartic superpotential

$$W \sim \mathrm{Tr}\,(A_i B_j A_k B_\ell)\, \epsilon^{ik}\epsilon^{j\ell}\,. \tag{1.1}$$

When $P=0$ above theory flows in the infrared to a superconformal fixed point, commonly referred to as Klebanov-Witten (KW) theory [1]. At the IR fixed point KW gauge theory is strongly coupled — the superconformal symmetry together with $SU(2)\times SU(2)\times U(1)$ global symmetry of the theory implies that anomalous dimensions of chiral superfields $\gamma(A_i)=\gamma(B_i)=-\frac{1}{4}$, *i.e.,* non-perturbatively large.

When $P\neq 0$, conformal invariance of the above $SU(K+P)\times SU(K)$ gauge theory is broken. It is useful to consider an effective description of this theory at energy scale μ with perturbative couplings $g_i(\mu)\ll 1$. It is straightforward to evaluate NSVZ beta-functions for the gauge couplings. One finds that while the sum of the gauge couplings does not run

$$\frac{d}{d\ln\mu}\left(\frac{\pi}{g_s}\equiv\frac{4\pi}{g_1^2(\mu)}+\frac{4\pi}{g_2^2(\mu)}\right)=0\,, \tag{1.2}$$

the difference between the two couplings is

$$\frac{4\pi}{g_2^2(\mu)}-\frac{4\pi}{g_1^2(\mu)}\sim P\ \left[3+2(1-\gamma_{ij})\right]\ \ln\frac{\mu}{\Lambda}\,, \tag{1.3}$$

where Λ is the strong coupling scale of the theory and γ_{ij} is an anomalous dimension of operators $\mathrm{Tr}\,A_iB_j$. Given (1.3) and (1.2) it is clear that the effective weakly coupled description of $SU(K+P)\times SU(K)$ gauge theory can be valid only in a finite-width energy band centered about μ scale. Indeed, extending effective description both to the UV and to the IR one necessarily encounters strong coupling in one or the other gauge group factor. As beautifully explained in [2], to extend the theory past the strongly coupled region(s) one must perform a Seiberg duality [3]. Turns out, in this gauge theory, a Seiberg duality transformation is a self-similarity transformation of the

effective description so that $K \to K - P$ as one flows to the IR, or $K \to K + P$ as the energy increases. Thus, extension of the effective $SU(K+P) \times SU(K)$ description to all energy scales involves and infinite sequence - a *cascade* - of Seiberg dualities where the rank of the gauge group is not constant along RG flow, but changes with energy according to [4–6]

$$K = K(\mu) \sim 2P^2 \ln \frac{\mu}{\Lambda}\,, \tag{1.4}$$

at least as $\mu \gg \Lambda$. To see (1.4), note that the rank changes by $\Delta K \sim P$ as $P\Delta\left(\ln \frac{\mu}{\Lambda}\right) \sim 1$. Although there are infinitely many duality cascade steps in the UV, there is only a finite number of duality transformations as one flows to the IR (from a given scale μ). The space of vacua of a generic cascading gauge theory was studied in details in [7]. In the simplest case, when $K(\mu)$ is an integer multiple of P, the cascading gauge theory confines in the infrared with a spontaneous breaking of the chiral symmetry [2].

Effective description of the cascading gauge theory in the UV suggests that it must be ultimately defined as a theory with an infinite number of degree of freedom. If so, an immediate concern is whether such a theory is renormalizable as a four dimensional quantum field theory, *i.e.,* whether a definite prescription can be made for the computation of all gauge invariant correlation functions in the theory. As was pointed out in [2], whenever $g_s K(\mu) \gg 1$, the cascading gauge theory allows for a dual holographic description [8, 9] as type IIB supergravity on a warped deformed conifold with fluxes. The duality is always valid in the UV of the cascading gauge theory; if, in addition, $g_s P \gg 1$ the holographic correspondence is valid in the IR as well. It was shown in [10] that a cascading gauge theory *defined* by its holographic dual as an RG flow of type IIB supergravity on a warped deformed conifold with fluxes is holographically renormalizable as a four dimensional quantum field theory.

Cascading gauge theories provide a soluble realization of the holographic gauge theory/string theory duality in non-conformal setting. A way to construct four dimensional examples of non-conformal gauge theory/string theory correspondence is to start with an AdS_5/CFT_4 duality and to deform it by relevant operators of the CFT_4. An example of such construction is the gauge/string duality for the $\mathcal{N} = 2^*$ supersymmetric gauge theory [11–13]. On the contrary, the scale in a cascading gauge theory is introduced via a dimensional transmutation of the gauge couplings (1.3). We would like to understand in details the hydrodynamic properties of strongly coupled non-conformal gauge theory plasmas [14]. A lot is known about thermodynamics/hydrodynamics of strongly coupled mass deformed conformal gauge theories from the perspective of

gauge/string correspondence [15–19]. The thermodynamic/hydrodynamic analysis of the cascading gauge theory plasma are substantially more difficult. The equilibrium thermodynamics of the cascading gauge theory plasma in the deconfined chirally symmetric phase is well understood by now [20–22]. Since at zero temperature a cascading gauge theory confines with a chiral symmetry breaking, it is conceivable that there is a finite-temperature deconfined phase of the theory, with broken chiral symmetry[1]. Whether or not such a phase exists is an open question [24]. In case of hydrodynamic transport coefficients[2], the shear viscosity was shown to satisfy the universal bound [26–28], and the bulk viscosity was computed to leading order at high temperature [29].

In this paper we study propagation of sound waves in the strongly coupled deconfined chirally symmetric phase of the cascading gauge theory plasma. In the dual gravitational description this involves computation of the dispersion relation of the lowest quasinormal mode in the sound channel [30] of the black hole solution numerically constructed in [22]. Ideally, we would like to do the analysis at any temperatures (at least above the deconfinement transition), much like it was done for the $\mathcal{N}=2^*$ plasma in [19]. Unfortunately, in section 3 we show that technical difficulties in the present framework does not allow us to achieve that goal. Thus, we resort to perturbative high temperature computations. The small parameter of the high temperature expansion is $\frac{P^2}{K_\star}$ [21], where $K_\star \simeq K(T)$ is roughly[3] the effective rank of the cascading plasma (1.4) at the IR cutoff scale, set by the temperature. We compute both the speed of the sound waves and the bulk viscosity of the cascading gauge theory plasma to order $\mathcal{O}\left(\frac{P^8}{K_\star^4}\right)$. Previously such analysis were done to order $\mathcal{O}\left(\frac{P^2}{K_\star}\right)$ [29]. At the order reported, the bulk viscosity was shown to saturate the bound proposed in [31]. Higher order corrections to the bulk viscosity of the cascading plasma presented here show that the bound [31] is satisfied. Alternative way to compute the speed of sound is to use the equilibrium equation of state

$$c_s^2 = \frac{\partial \mathcal{P}}{\partial \mathcal{E}}\,. \tag{1.5}$$

Using (1.5) and slightly extending the computations in [22], we can evaluate the speed of sound to temperatures down to the deconfinement transition and below. The com-

[1]Such a phase was observed in $4+1$ dimensional supersymmetric $SU(N_c)$ gauge theory with fundamental quarks compactified on a circle [23].

[2]Hydrodynamics of closely related models was recently discussed in [25].

[3]A precise definition of $K_\star$ is given below.

parison between the perturbative high temperature analysis and the exact one indicates that the former is convergent for $\frac{K_\star}{P^2} \sim 2 \cdots 3$, correspondingly to temperatures $T \simeq (1 \cdots 1.5)\Lambda$ — which is about twice the critical temperature of the deconfinement phase transition $T_{critical} = 0.6141111(3)\Lambda$ [22]. Thus, although we find a relatively small bulk viscosity in the high temperature regime $\frac{\zeta}{\eta} \lesssim \frac{1}{2}$, we can not (reliably) evaluate the bulk viscosity in the vicinity of the deconfinement transition. It it clear though that since the deconfinement phase transition is of the first-order (in the 't Hooft limit), the bulk viscosity will remain finite at the transition point [32]. Finally, we find that chirally symmetric phase becomes perturbatively unstable at $T = T_{unstable} = 0.87487(7)T_{critical}$ — exactly at this temperature c_s^2 vanishes, and extending this phase to lower values of $K_\star$ leads to $c_s^2 < 0$. The critical behavior at the unstable point in the cascading plasma is identical to the one found in $\mathcal{N} = 2^*$ plasma with mass deformation parameters $m_f < m_b$ [19, 31]. We comment more on the instability in section 5.

The technical aspects of the computations are presented in section 2. The reader interested only in the results should consult section 4.

2 Supergravity dual to deconfined cascading plasma

The holographic dual to a deconfined chirally symmetric phase of a cascading gauge theory plasma at equilibrium is given by a black hole solution in a singular Klebanov-Tseytlin (KT) geometry [33]. It has been discussed previously in [4, 20–22]. We follow the notations of [22]. A black hole metric[4] :

$$\begin{split} ds_{10}^2 =& h^{-1/2}(2x-x^2)^{-1/2}\left(-(1-x)^2dt^2+dx_1^2+dx_2^2+dx_3^2\right)+G_{xx}(dx)^2 \\ &+h^{1/2}[f_2\ \left(e_\psi^2\right)+f_3\ \sum_{a=1}^2\left(e_{\theta_a}^2+e_{\phi_a}^2\right)]\,, \end{split} \tag{2.1}$$

where h, f_2 and f_3 are some functions of the radial coordinate x. There is also a dilaton $g(x)$, and form fields given by

$$\begin{split} &F_3 = P\ e_\psi \wedge (e_{\theta_1}\wedge e_{\phi_1} - e_{\theta_2}\wedge e_{\phi_2})\,, \qquad B_2 = \frac{K}{2P}\ (e_{\theta_1}\wedge e_{\phi_1} - e_{\theta_2}\wedge e_{\phi_2})\,, \\ &F_5 = \mathcal{F}_5 + \star\mathcal{F}_5\,, \qquad \mathcal{F}_5 = -K\ e_\psi\wedge e_{\theta_1}\wedge e_{\phi_1}\wedge e_{\theta_2}\wedge e_{\phi_2}\,, \end{split} \tag{2.2}$$

[4] The frames $\{e_{\theta_a}, e_{\phi_a}\}$ are defined as in [10], such that the metric on a unit size $T^{1,1}$ is given by $\left(e_\psi^2\right) + \sum_{a=1}^2 \left(e_{\theta_a}^2 + e_{\phi_a}^2\right)$.

where K is a function of the radial coordinate x. Without loss of generality we can set asymptotic string coupling to one. We use the following parametrization for the solution in perturbation theory in $\frac{P^2}{K_\star}$:

$$h(x) = \frac{K_\star}{4\tilde{a}_0^2} + \frac{K_\star}{\tilde{a}_0^2} \sum_{n=1}^{\infty} \left\{ \left(\frac{P^2}{K_\star}\right)^n \left(\xi_{2n}(x) - \frac{5}{4}\eta_{2n}(x)\right) \right\} , \tag{2.3}$$

$$f_2(x) = \tilde{a}_0 + \tilde{a}_0 \sum_{n=1}^{\infty} \left\{ \left(\frac{P^2}{K_\star}\right)^n \left(-2\xi_{2n}(x) + \eta_{2n}(x) + \frac{4}{5}\lambda_{2n}(x)\right) \right\} , \tag{2.4}$$

$$f_3(x) = \tilde{a}_0 + \tilde{a}_0 \sum_{n=1}^{\infty} \left\{ \left(\frac{P^2}{K_\star}\right)^n \left(-2\xi_{2n}(x) + \eta_{2n}(x) - \frac{1}{5}\lambda_{2n}(x)\right) \right\} , \tag{2.5}$$

$$K(x) = K_\star + K_\star \sum_{n=1}^{\infty} \left\{ \left(\frac{P^2}{K_\star}\right)^n \kappa_{2n}(x) \right\} , \tag{2.6}$$

$$g(x) = 1 + \sum_{n=1}^{\infty} \left\{ \left(\frac{P^2}{K_\star}\right)^n \zeta_{2n}(x) \right\} . \tag{2.7}$$

The advantage of this parametrization is that the equations for $\{\xi_{2n}, \eta_{2n}, \lambda_{2n}, \zeta_{2n}\}$ decouple, once the (decoupled) equation for κ_{2n} is solved, at each order n in perturbation theory.

Gauge invariant fluctuations

$$\{Z_H\,, Z_f\,, Z_\omega\,, Z_\Phi\,, Z_K\}$$

of the background metric (2.1) and the scalar fields

$$\left\{ f \equiv \frac{1}{4}\ln h + \frac{2}{5}\ln f_3 + \frac{1}{10}\ln f_2\,, \qquad \omega = \frac{1}{10}\ln\frac{f_3}{f_2}\,, \qquad \Phi = \ln g\,, \qquad K \right\}$$

of the effective five-dimensional gravitational description of the sound channel quasi-normal modes were studied in details in [29][5]. The incoming wave boundary conditions on all physical modes imply that

$$\begin{aligned} &Z_H(x) = (1-x)^{-i\mathfrak{w}} z_H(x)\,, \qquad Z_f(x) = (1-x)^{-i\mathfrak{w}} z_f(x)\,, \\ &Z_w(x) = (1-x)^{-i\mathfrak{w}} z_w(x)\,, \qquad Z_\Phi(x) = (1-x)^{-i\mathfrak{w}} z_\Phi(x)\,, \\ &Z_K(x) = (1-x)^{-i\mathfrak{w}} z_K(x)\,, \end{aligned} \tag{2.8}$$

[5]See eq. (63) of [29] for the definition of the gauge invariant fluctuations and eqs. (64)-(68) of [29] for the corresponding equations of motion.

where $\{z_H, z_f, z_w, z_\Phi, z_K\}$ are regular at the horizon; we further introduced

$$\mathfrak{w} \equiv \frac{\omega}{2\pi T}\,, \qquad \mathfrak{q} \equiv \frac{q}{2\pi T}\,. \tag{2.9}$$

T is the equilibrium temperature of the plasma, and $\{\omega, q = |\vec{q}|\}$ are the frequency and the momentum of the sound quasinormal mode. There is a single integration constant for these physical modes, namely, the overall scale. Without the loss of generality the latter can be fixed as

$$z_H(x)\bigg|_{x\to 1_-} = 1\,. \tag{2.10}$$

In this case, the sound dispersion relation is simply determined as

$$z_H(x)\bigg|_{x\to 0_+} = 0\,. \tag{2.11}$$

The other boundary conditions (besides regularity at the horizon and (2.11)) are

$$z_f(x)\bigg|_{x\to 0_+} = 0\,,\ \ z_w(x)\bigg|_{x\to 0_+} = 0\,,\ \ z_\Phi(x)\bigg|_{x\to 0_+} = 0\,,\ \ z_K(x)\bigg|_{x\to 0_+} = 0\,. \tag{2.12}$$

Let's introduce

$$\begin{split}
z_H &= \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{H,0}^{(n)} \right\} + i\ \mathfrak{q}\ \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{H,1}^{(n)} \right\}\,,\\
z_f &= \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{f,0}^{(n)} \right\} + i\ \mathfrak{q}\ \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{f,1}^{(n)} \right\}\,,\\
z_\omega &= \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{\omega,0}^{(n)} \right\} + i\ \mathfrak{q}\ \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{\omega,1}^{(n)} \right\}\,,\\
z_\Phi &= \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{\Phi,0}^{(n)} \right\} + i\ \mathfrak{q}\ \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{\Phi,1}^{(n)} \right\}\,,\\
z_K &= K_\star \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{K,0}^{(n)} \right\} + i\ \mathfrak{q}\ K_\star \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n\ z_{K,1}^{(n)} \right\}\,,
\end{split} \tag{2.13}$$

where the lower index refers to either the leading, $\propto \mathfrak{q}^0$, or to the next-to-leading, $\propto \mathfrak{q}^1$, order in the hydrodynamic approximation, and the upper index keeps track of the $\frac{P^2}{K_\star}$ high temperature expansion parameter. Additionally, we find it convenient to parametrize

$$\mathfrak{w} = \frac{\mathfrak{q}}{\sqrt{3}} \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n \beta_{1,n} \right\} - i\ \frac{\mathfrak{q}^2}{3} \sum_{n=0}^\infty \left\{ \left(\frac{P^2}{K_\star}\right)^n \beta_{2,n} \right\}\,, \tag{2.14}$$

where $\beta_{1,n}$, $\beta_{2,n}$ are constants which are to be determined from the pole dispersion relation (2.11)

$$
z_{H,0}^{(n)}\bigg|_{x\to 0_+}=0\,,\qquad z_{H,1}^{(n)}\bigg|_{x\to 0_+}=0\,. \tag{2.15}
$$

2.1 Equilibrium thermodynamics to order $\mathcal{O}\left(\frac{P^8}{K_\star^4}\right)$

Cascading gauge theory thermodynamics for generic $\frac{P^2}{K_\star}$ was studied in [22]. Since we are able to compute the $\mathcal{O}(\mathfrak{q}^2)$ sound wave dispersion relation only perturbatively in $\frac{P^2}{K_\star}$, we need to construct the black hole backgrounds in the high temperature expansion as well. To order $\mathcal{O}\left(\frac{P^2}{K_\star}\right)$ this was done in [21], and extended to order $\mathcal{O}\left(\frac{P^6}{K_\star^3}\right)$ in [22]. Here, we extend the analysis to $n=4$ in (2.3)-(2.7).

Equations of motion for $\{\kappa_{2n}\,,\xi_{2n}\,,\eta_{2n}\,,\lambda_{2n}\,,\zeta_{2n}\}$ take form

$$
0=\kappa_{2n}''+\frac{\kappa_{2n}'}{x-1}+\mathcal{J}_{b,\kappa}^{[2n]}\,, \tag{2.16}
$$

$$
0=\eta_{2n}''+\frac{\eta_{2n}'}{x-1}-\frac{8\eta_{2n}}{x^2(x-2)^2}-\frac{2}{5}\kappa_2'\ \kappa_{2n}'-\frac{8\kappa_{2n}}{3x^2(x-2)^2}+\mathcal{J}_{b,\eta}^{[2n]}\,, \tag{2.17}
$$

$$
0=\xi_{2n}''+\frac{(3x^2-6x+4)\xi_{2n}'}{x(x-1)(x-2)}-\frac{2}{3}\kappa_2'\ \kappa_{2n}'+\mathcal{J}_{b,\xi}^{[2n]}\,, \tag{2.18}
$$

$$
0=\lambda_{2n}''+\frac{\lambda_{2n}'}{x-1}-\frac{3\lambda_{2n}}{x^2(x-2)^2}-2\kappa_2'\ \kappa_{2n}'+\mathcal{J}_{b,\lambda}^{[2n]}\,, \tag{2.19}
$$

$$
0=\zeta_{2n}''+\frac{\zeta_{2n}'}{x-1}+2\kappa_2'\ \kappa_{2n}'+\mathcal{J}_{b,\zeta}^{[2n]}\,, \tag{2.20}
$$

where the source terms $\{\mathcal{J}_{b,\kappa}^{[2n]},\mathcal{J}_{b,\eta}^{[2n]},\mathcal{J}_{b,\xi}^{[2n]},\mathcal{J}_{b,\lambda}^{[2n]},\mathcal{J}_{b,\zeta}^{[2n]}\}$ are functionals of the lower order solutions: κ_{2m}, ξ_{2m}, η_{2m}, λ_{2m}, ζ_{2m}, with $m<n$. Explicit expressions for the source term functionals are available from the author upon request. The perturbative solutions to (2.16)-(2.20) must be regular at the horizon, and must have the appropriate KT asymptotics near the boundary. Beyond $n=1$, these equations must be solved numerically. We apply the numerical strategy developed in [22]:

▪ Generically, the differential equations will have non-normalizable modes near the boundary[6] $x\to 0_+$, and can generate singular Schwarzschild horizon as $x\to 1_-$. Thus,

[6] These modes are singular in case they are dual to operators of dimension larger than four — as for $\{f,\omega\}$; in other cases they modify the KT asymptotics, *i.e.,* the parameters (such as a strong coupling scale) of the dual plasma.

we specify boundary conditions as a series expansion near the boundary and the horizon which explicitly contain only normalizable modes.

- The total number of integration constants near the boundary and the horizon appearing in normalizable modes precisely equals the total order of the system of ODE's. As a result the boundary value problem is well posed.
- We solve the resulting boundary value problem as detailed in section 5.2 of [22].

In what follows we present the horizon and the boundary expansion of the normalizable modes for $\{\kappa_{2n}\,,\xi_{2n}\,,\eta_{2n}\,,\lambda_{2n}\,,\zeta_{2n}\}$ with $n=\{1,2,3,4\}$. In numerical analysis we used expansion to order $\mathcal{O}(x^{9/2})$ (up to powers of $\ln x$) near the boundary; and to order $\mathcal{O}(y^{10})$, $y=1-x$, near the horizon. Below, however, we present expansion at most to orders $\mathcal{O}(x^2)$ and $\mathcal{O}(y^2)$ — just what is enough to exhibit the dependence on all the integration constants. We solve the boundary value problem on the interval $x\in[\delta_x,1-\delta_x]$ with $\delta_x=10^{-2}$; we verified that our final results are insensitive to the precise choice of δ_x, provided it is sufficiently small.

Notice from (2.18) that ξ_{2n} always has a zero mode. Such a zero mode simply rescales (perturbatively in $\frac{P^2}{K_\star}$) $\tilde{a}_0$, and has no effect on physical quantities [22]. In what follows we conveniently set this mode to zero near the boundary[7] — for further details see (2.31), (2.42), (2.53) below. Naively, from (2.20), ζ_{2n} also always has a zero mode. The latter however is fixed by our choice of the asymptotic string coupling (2.7). Similarly, the zero mode of κ_{2n}, see (2.16), modifies the strong coupling scale Λ of the cascading plasma.

2.1.1 *Order* $n=1$

We find:

$$\kappa_2=-\frac{1}{2}\ln(2x-x^2)\,,\tag{2.21}$$

$$\xi_2=\frac{1}{12}\ln(2x-x^2)\,.\tag{2.22}$$

Even though it is possible to write down explicit analytic expressions for $\{\eta_2,\lambda_2,\zeta_2\}$, such expressions involve complicated polylogarithm functions, which slows down subsequent numerical computations. Thus, we opt to treat these fields numerically.

[7] Of course, it is inconsistent to require the vanishing of the zero mode both near the boundary and the horizon.

Near the boundary, $x \to 0_+$, we have:

$$\eta_2 = -\frac{1}{6} + \frac{1}{6}\ln 2 + \frac{1}{6}\ln x - \frac{1}{30}x + x^2\left(\eta_1^{4,0} + \frac{1}{30}\ln x\right) + \mathcal{O}(x^3 \ \ln x)\,, \tag{2.23}$$

$$\lambda_2 = \frac{2}{3}x + \lambda_1^{3,0}x^{3/2} + \mathcal{O}(x^2)\,, \tag{2.24}$$

$$\zeta_2 = x\left(\zeta_1^{2,0} + \frac{1}{2}\ln x\right) + \mathcal{O}(x^2 \ \ln x)\,. \tag{2.25}$$

Near the horizon, $y \to 0_+$, we have:

$$\eta_2 = \eta_{1,h}^0 + \mathcal{O}(y^2)\,, \tag{2.26}$$

$$\lambda_2 = \lambda_{1,h}^0 + \mathcal{O}(y^2)\,, \tag{2.27}$$

$$\zeta_2 = \zeta_{1,h}^0 + \mathcal{O}(y^2)\,. \tag{2.28}$$

Altogether at this order we have 6 integration constants

$$\{\eta_1^{4,0}\,, \lambda_1^{3,0}\,, \zeta_1^{2,0}\,, \eta_{1,h}^0\,, \lambda_{1,h}^0\,, \zeta_{1,h}^0\}\,, \tag{2.29}$$

which is precisely what is needed to specify a unique solution for $\{\eta_2, \lambda_2, \zeta_2\}$.

2.1.2 Order $n = 2$

Near the boundary, $x \to 0_+$, we have:

$$\kappa_4 = x\left(\kappa_2^{2,0} - \frac{1}{2}\ln x\right) + \mathcal{O}(x^{3/2})\,, \tag{2.30}$$

$$\xi_4 = \frac{1}{36}\ln x + \mathcal{O}(x \ \ln x)\,, \tag{2.31}$$

$$\begin{aligned}\eta_4 = &-\frac{1}{12} + \frac{1}{18}\ln 2 + \frac{1}{18}\ln x + x\left(-\frac{1}{30}\zeta_1^{2,0} + \frac{7}{360} - \frac{1}{60}\ln 2 - \frac{7}{30}\kappa_2^{2,0} + \frac{1}{12}\ln x\right)\\ &- \frac{4}{225}\lambda_1^{3,0}x^{3/2} + x^2\left(\eta_2^{4,0} + \left(-\frac{4}{3}\eta_1^{4,0} + \frac{1}{18}\zeta_1^{2,0} - \frac{7}{1080} - \frac{1}{90}\ln 2 + \frac{1}{15}\kappa_2^{2,0}\right)\ln x\right.\\ &\left.- \frac{11}{360}\ln^2 x\right) + \mathcal{O}(x^{5/2})\,,\end{aligned} \tag{2.32}$$

$$\lambda_4 = x\left(\frac{4}{3}\kappa_2^{2,0} - \frac{11}{9} + \frac{2}{3}\zeta_1^{2,0} + \frac{1}{3}\ln 2\right) + x^{3/2}\lambda_2^{3,0} + \mathcal{O}(x^2 \ \ln x)\,, \tag{2.33}$$

$$\zeta_4 = x\left(\zeta_2^{2,0} + \left(\kappa_2^{2,0} - \frac{5}{6} + \frac{1}{4}\ln 2 + \frac{1}{2}\zeta_1^{2,0}\right)\ln x\right) + \mathcal{O}(x^2\ \ln^2 x)\,. \tag{2.34}$$

Near the horizon, $y \to 0_+$, we have:

$$\kappa_4 = \kappa_{2,h}^0 + \mathcal{O}(y^2)\,, \tag{2.35}$$

$$\xi_4 = \xi_{2,h}^0 + \xi_{2,h}^1 y^2 + \mathcal{O}(y^4)\,, \tag{2.36}$$

$$\eta_4 = \eta_{2,h}^0 + \mathcal{O}(y^2)\,, \tag{2.37}$$

$$\lambda_4 = \lambda_{2,h}^0 + \mathcal{O}(y^2)\,, \tag{2.38}$$

$$\zeta_4 = \zeta_{2,h}^0 + \mathcal{O}(y^2)\,. \tag{2.39}$$

Altogether at this order we have 10 integration constants

$$\{\kappa_2^{2,0}\,, \eta_2^{4,0}\,, \lambda_2^{3,0}\,, \zeta_2^{2,0}\,, \kappa_{2,h}^0\,, \xi_{2,h}^0\,, \xi_{2,h}^1\,, \eta_{2,h}^0\,, \lambda_{2,h}^0\,, \zeta_{2,h}^0\}\,, \tag{2.40}$$

which is precisely what is needed to specify a unique solution for $\{\kappa_4, \xi_4, \eta_4, \lambda_4, \zeta_4\}$.

2.1.3 Order $n = 3$

Near the boundary, $x \to 0_+$, we have:

$$\kappa_6 = x\left(\kappa_3^{2,0} + \left(-\kappa_2^{2,0} + \frac{5}{6} - \frac{1}{4}\ln 2 - \frac{1}{2}\zeta_1^{2,0}\right)\ln x\right) + \mathcal{O}(x^{3/2})\,, \tag{2.41}$$

$$\xi_6 = \left(-\frac{1}{108}\ln 2 + \frac{1}{48}\right)\ln x + \mathcal{O}(x\ \ln x)\,, \tag{2.42}$$

$$\begin{aligned}
\eta_6 = & -\frac{1}{54}\ln^2 2 + \frac{17}{216}\ln 2 - \frac{49}{648} + \left(-\frac{1}{54}\ln 2 + \frac{1}{24}\right)\ln x + x\Bigg(-\frac{47}{1080} - \frac{1}{60}\zeta_1^{2,0}\ln 2 \\
& - \frac{1}{30}\kappa_2^{2,0}\ln 2 + \frac{77}{2160}\ln 2 + \frac{7}{360}\zeta_1^{2,0} - \frac{1}{60}\kappa_2^{2,0} - \frac{1}{30}\zeta_2^{2,0} - \frac{7}{30}\kappa_3^{2,0} - \frac{1}{120}\ln^2 2 \\
& + \left(\frac{1}{6}\kappa_2^{2,0} + \frac{1}{12}\zeta_1^{2,0} + \frac{1}{24}\ln 2 - \frac{1}{9}\right)\ln x\Bigg) + x^{3/2}\left(-\frac{2}{225}\lambda_1^{3,0} - \frac{4}{225}\lambda_2^{3,0}\right) \\
& + x^2\Bigg(\eta_3^{4,0} + \Bigg(\frac{1}{9}\zeta_1^{2,0}\kappa_2^{2,0} + \frac{1}{15}(\kappa_2^{2,0})^2 - \frac{1}{45}\kappa_2^{2,0}\ln 2 + \frac{89}{72}\eta_1^{4,0} - \frac{19}{1440}\ln 2 - \frac{163}{1620}\zeta_1^{2,0} \\
& - \frac{17}{90}\kappa_2^{2,0} + \frac{25121}{259200} - \frac{17}{18}\eta_1^{4,0}\ln 2 - \frac{4}{3}\eta_2^{4,0} + \frac{37}{540}(\zeta_1^{2,0})^2 + \frac{1}{18}\zeta_2^{2,0} + \frac{1}{15}\kappa_3^{2,0} \\
& - \frac{1}{180}\ln^2 2\Bigg)\ln x + \left(-\frac{1}{15}\kappa_2^{2,0} + \frac{5}{12}\eta_1^{4,0} - \frac{1}{30}\zeta_1^{2,0} - \frac{7}{720}\ln 2 + \frac{421}{8640}\right)\ln^2 x \\
& + \frac{1}{144}\ln^3 x\Bigg) + \mathcal{O}(x^{5/2}\ \ln x)\,,
\end{aligned} \tag{2.43}$$

$$\lambda_6 = x\bigg(-\frac{61}{54}\ln 2 + \frac{107}{54} + \frac{1}{3}\zeta_1^{2,0}\ln 2 + \frac{2}{3}\kappa_2^{2,0}\ln 2 + \frac{1}{6}\ln^2 2 - \frac{11}{9}\zeta_1^{2,0} - \frac{22}{9}\kappa_2^{2,0} + \frac{2}{3}\zeta_2^{2,0}$$
$$+ \frac{4}{3}\kappa_3^{2,0}\bigg) + x^{3/2}\lambda_3^{3,0} + \mathcal{O}(x^2\ \ln x)\,, \tag{2.44}$$

$$\zeta_6 = x\bigg(\zeta_3^{2,0} + \bigg(-\frac{5}{6}\zeta_1^{2,0} - \frac{5}{3}\kappa_2^{2,0} + \frac{1}{2}\zeta_2^{2,0} + \kappa_3^{2,0} + \frac{25}{18} - \frac{5}{6}\ln 2 + \frac{1}{8}\ln^2 2 + \frac{1}{4}\zeta_1^{2,0}\ln 2$$
$$+ \frac{1}{2}\kappa_2^{2,0}\ln 2\bigg)\ln x\bigg) + \mathcal{O}(x^2\ \ln^2 x)\,. \tag{2.45}$$

Near the horizon, $y \to 0_+$, we have:

$$\kappa_6 = \kappa_{3,h}^0 + \mathcal{O}(y^2)\,, \tag{2.46}$$

$$\xi_6 = \xi_{3,h}^0 + \xi_{3,h}^1 y^2 + \mathcal{O}(y^4)\,, \tag{2.47}$$

$$\eta_6 = \eta_{3,h}^0 + \mathcal{O}(y^2)\,, \tag{2.48}$$

$$\lambda_6 = \lambda_{3,h}^0 + \mathcal{O}(y^2)\,, \tag{2.49}$$

$$\zeta_6 = \zeta_{3,h}^0 + \mathcal{O}(y^2)\,. \tag{2.50}$$

Altogether at this order we have 10 arbitrary integration constants

$$\{\kappa_3^{2,0}\,, \eta_3^{4,0}\,, \lambda_3^{3,0}\,, \zeta_3^{2,0}\,, \kappa_{3,h}^0\,, \xi_{3,h}^0\,, \xi_{3,h}^1\,, \eta_{3,h}^0\,, \lambda_{3,h}^0\,, \zeta_{3,h}^0\}\,, \tag{2.51}$$

which is precisely what is needed to specify a unique solution for $\{\kappa_6, \xi_6, \eta_6, \lambda_6, \zeta_6\}$.

2.1.4 Order $n = 4$

Ultimately, to evaluate the speed of sound and the bulk viscosity at this order we will need only κ_8 and ξ_8 solutions.

Near the boundary, $x \to 0_+$, we have:

$$\kappa_8 = x\bigg(\kappa_4^{2,0} + \bigg(\frac{5}{6}\zeta_1^{2,0} + \frac{5}{3}\kappa_2^{2,0} - \frac{1}{2}\zeta_2^{2,0} - \kappa_3^{2,0} - \frac{25}{18} + \frac{5}{6}\ln 2 - \frac{1}{8}\ln^2 2 - \frac{1}{4}\zeta_1^{2,0}\ln 2$$
$$- \frac{1}{2}\kappa_2^{2,0}\ln 2\bigg)\ln x\bigg) + \mathcal{O}(x^{3/2})\,, \tag{2.52}$$

n	1	2	3	4
$\kappa_n^{2,0}$		0.73675974	-0.62226255	-0.03784377
$\eta_n^{4,0}$	-0.01717287	0.00534036	-0.01064222	
$\lambda_n^{3,0}$	-0.87235794	-1.11562943	1.39008636	
$\zeta_n^{2,0}$	-0.15342641	0.62226267	-0.32514260	
$\kappa_{n,h}^0$		0.62226259	-0.42061461	0.00816831
$\xi_{n,h}^0$		-0.07981931	0.01661150	-0.00920379
$\xi_{n,h}^0$		0.01919989	-0.05277626	0.01385333
$\eta_{n,h}^0$	-0.14891337	-0.21809464	0.00213345	
$\lambda_{n,h}^0$	0.16806881	-0.14619173	0.01639579	
$\zeta_{n,h}^0$	-0.41123352	0.33024116	-0.07445122	

Table 1: Coefficients of the normalizable modes of the background geometry. See (2.29), (2.40), (2.51) and (2.56).

$$\xi_8 = \left(\frac{1}{324}\ln^2 2 - \frac{23}{1296}\ln 2 + \frac{41}{1944}\right)\ln x + \mathcal{O}(x\ \ln x)\,. \tag{2.53}$$

Near the horizon, $y \to 0_+$, we have:

$$\kappa_8 = \kappa_{4,h}^0 + \mathcal{O}(y^2)\,, \tag{2.54}$$

$$\xi_8 = \xi_{4,h}^0 + \xi_{4,h}^1 y^2 + \mathcal{O}(y^4)\,. \tag{2.55}$$

Altogether at this order we have 4 integration constants

$$\{\kappa_4^{2,0}\,, \kappa_{4,h}^0\,, \xi_{4,h}^0\,, \xi_{4,h}^1\}\,, \tag{2.56}$$

which is precisely what is needed to specify a unique solution for $\{\kappa_8, \xi_8\}$.

2.1.5 Integration constants for the normalizable modes

Here we tabulate (see table 1) the integration constants for the normalizable modes of κ_{2n}, ξ_{2n}, η_{2n}, λ_{2n}, ζ_{2n} with $n = \{1,2,3,4\}$ obtained from solving the corresponding boundary value problems.

2.1.6 $\mathcal{P}$, $\mathcal{E}$ and c_s^2 from equilibrium thermodynamics

Using the results of [22] we can compute

$$\frac{\mathcal{P}}{sT} = \frac{3}{7}\left(\frac{7}{12} - \hat{a}_{2,0}\right), \qquad \frac{\mathcal{E}}{sT} = \frac{3}{4}\left(1 + \frac{4}{7}\,\hat{a}_{2,0}\right), \tag{2.57}$$

where s is the entropy density[8]. Furthermore, the perturbative high temperature expansion for $\hat{a}_{2,0}$ is given by[9]

$$\begin{aligned}
\hat{a}_{2,0} =& \frac{7}{12}\frac{P^2}{K_\star} + \left(\frac{7}{6}\kappa_2^{2,0} - \frac{35}{3} + \frac{7}{12}\zeta_1^{2,0} + \frac{7}{24}\ln 2\right)\frac{P^4}{K_\star^2} + \left(-\frac{35}{36}\ln 2 - \frac{35}{18}\kappa_2^{2,0} + \frac{7}{48}\ln^2 2\right. \\
&+ \left.\frac{175}{108} + \frac{7}{24}\zeta_1^{2,0}\ln 2 + \frac{7}{12}\kappa_2^{2,0}\ln 2 + \frac{7}{12}\zeta_2^{2,0} + \frac{7}{6}\kappa_3^{2,0} - \frac{35}{36}\zeta_1^{2,0}\right)\frac{P^6}{K_\star^3} + \left(\frac{175}{72}\ln 2\right. \\
&+ \frac{7}{12}\zeta_3^{2,0} + \frac{175}{54}\kappa_2^{2,0} - \frac{35}{18}\kappa_2^{2,0}\ln 2 - \frac{35}{36}\zeta_1^{2,0}\ln 2 - \frac{875}{324} + \frac{175}{108}\zeta_1^{2,0} - \frac{35}{48}\ln^2 2 \\
&+ \frac{7}{96}\ln^3 2 + \frac{7}{48}\zeta_1^{2,0}\ln^2 2 + \frac{7}{12}\kappa_3^{2,0}\ln 2 + \frac{7}{24}\kappa_2^{2,0}\ln^2 2 + \frac{7}{24}\zeta_2^{2,0}\ln 2 + \frac{7}{6}\kappa_4^{2,0} \\
&- \left.\frac{35}{36}\zeta_2^{2,0} - \frac{35}{18}\kappa_3^{2,0}\right)\frac{P^8}{K_\star^4} + \mathcal{O}\left(\frac{P^{10}}{K_\star^5}\right).
\end{aligned} \tag{2.58}$$

Note that the coefficient of $\frac{P^4}{K_\star^2}$ must vanish [22] — numerically we find that it is $\propto 2\times 10^{-10}$.

The precise temperature dependence of $K_\star$ was determined in [22]

$$\frac{K_\star}{P^2} = \frac{1}{2}\ln\left(\frac{64\pi^4}{81} \;\times\; \frac{sT}{\Lambda^4}\right). \tag{2.59}$$

Using (2.59) and the expressions for the pressure and the energy density from (2.57) we find

$$c_s^2 = \frac{\partial \mathcal{P}}{\partial \mathcal{E}} = \frac{1}{3}\,\frac{7 - 12\hat{a}_{2,0} - 6P^2\,\frac{d\hat{a}_{2,0}}{dK_\star}}{7 + 4\hat{a}_{2,0} + 2P^2\,\frac{d\hat{a}_{2,0}}{dK_\star}}, \tag{2.60}$$

Thus, given the perturbative high temperature expansion for $\hat{a}_{2,0}$ we can evaluate from

[8] Note that expressions in (2.57) are valid for any temperature.

[9] In order to evaluate the coefficient of $\frac{P^8}{K_\star^4}$ term we need the boundary expansions for η_4 and λ_4 to order $\mathcal{O}(x)$. These expansions do not depend on the coefficients of the corresponding normalizable modes $\{\eta_4^{4,0}, \lambda_4^{3,0}\}$.

(2.60) the perturbative high temperature expansion for c_s^2

$$
\begin{aligned}
3c_s^2 = &= 1 - \frac{4}{3}\frac{P^2}{K_\star} + \left(\frac{10}{3} - \frac{2}{3}\ln 2 - \frac{8}{3}\kappa_2^{2,0} - \frac{4}{3}\zeta_1^{2,0}\right)\frac{P^4}{K_\star^2} + \left(-8 - \frac{2}{3}\zeta_1^{2,0}\ln 2\right.\\
&\left. + \frac{10}{3}\ln 2 - \frac{1}{3}\ln^2 2 - \frac{8}{3}\kappa_3^{2,0} + \frac{40}{9}\zeta_1^{2,0} - \frac{4}{3}\kappa_2^{2,0}\ln 2 + \frac{80}{9}\kappa_2^{2,0} - \frac{4}{3}\zeta_2^{2,0}\right)\frac{P^6}{K_\star^3}\\
&+ \left(-\frac{1}{3}\ln^2 2\zeta_1^{2,0} - \frac{4}{3}\kappa_3^{2,0}\ln 2 - \frac{2}{3}\kappa_2^{2,0}\ln^2 2 - \frac{2}{3}\zeta_2^{2,0}\ln 2 + \frac{16}{9}(\kappa_2^{2,0})^2 + \frac{16}{9}\kappa_2^{2,0}\zeta_1^{2,0}\right.\\
&+ \frac{4}{9}(\zeta_1^{2,0})^2 - \frac{1}{6}\ln^3 2 - 12\ln 2 + \frac{37}{9}\zeta_1^{2,0}\ln 2 + \frac{169}{9} + \frac{74}{9}\kappa_2^{2,0}\ln 2 + \frac{5}{2}\ln^2 2\\
&\left. - \frac{212}{9}\kappa_2^{2,0} - \frac{106}{9}\zeta_1^{2,0} + \frac{46}{9}\zeta_2^{2,0} + \frac{92}{9}\kappa_3^{2,0} - \frac{4}{3}\zeta_3^{2,0} - \frac{8}{3}\kappa_4^{2,0}\right)\frac{P^8}{K_\star^4} + \mathcal{O}\left(\frac{P^{10}}{K_\star^5}\right)\,.
\end{aligned}
\tag{2.61}
$$

Notice that (2.61) provides *predictions* for $\beta_{1,n}$ of (2.14) with $n = 1, \cdots 4$.

2.2 Speed of sound waves in the cascading plasma to order $\mathcal{O}\left(\frac{P^8}{K_\star^4}\right)$

Equations of motion for the sound waves in the cascading plasma for generic $\frac{P^2}{K_\star}$ were derived in [29]. Previously, they have been discussed (solved) only to order $\mathcal{O}\left(\frac{P^2}{K_\star}\right)$, [29]. Here, we extend the analysis to $n = 4$ in (2.13) at order $\mathcal{O}(\mathfrak{q}^0)$.

Equations of motion for $\{z_{H,0}^{(n)}, z_{f,0}^{(n)}, z_{\omega,0}^{(n)}, z_{\Phi,0}^{(n)}, z_{K,0}^{(n)}\}$ take form[10]

$$
\begin{aligned}
0 = &\left[z_{H,0}^{(n)}\right]'' - \frac{3x^2 - 6x + 2}{(x-1)(x^2 - 2x + 2)}\ \left[z_{H,0}^{(n)}\right]' + \frac{4}{x^2 - 2x + 2}\ z_{H,0}^{(n)}\\
&+ \frac{32}{x^2 - 2x + 2}\ z_{K,0}^{(n)} - \frac{16x(2-x)}{3(1-x)}\ \kappa_{2n}' - \frac{16x^2(2-x)^2}{(x-1)(x^2 - 2x + 2)}\ \xi_{2n}'\\
&+ \frac{8\beta_{1,n}}{x^2 - 2x + 2} + \mathcal{J}_{s,H}^{[2n]}\,,
\end{aligned}
\tag{2.62}
$$

$$
\begin{aligned}
0 = &\left[z_{f,0}^{(n)}\right]'' + \frac{1}{x-1}\ \left[z_{f,0}^{(n)}\right]' - \frac{8}{x^2(2-x)^2}\ z_{f,0}^{(n)} + \frac{3(x-1)}{10x(2-x)}\ \left[z_{K,0}^{(n)}\right]'\\
&+ \frac{2}{x^2(2-x)^2}\ z_{K,0}^{(n)} + \frac{2}{3x(2-x)(1-x)^2}\ \kappa_{2n} + \mathcal{J}_{s,f}^{[2n]}\,,
\end{aligned}
\tag{2.63}
$$

$$
\begin{aligned}
0 = &\left[z_{\omega,0}^{(n)}\right]'' + \frac{1}{x-1}\ \left[z_{\omega,0}^{(n)}\right]' - \frac{3}{x^2(2-x)^2}\ z_{\omega,0}^{(n)} + \frac{(x-1)}{5x(2-x)}\ \left[z_{K,0}^{(n)}\right]'\\
&+ \frac{1}{15(1-x)^3}\ \lambda_{2n}' + \frac{1}{10x(2-x)(1-x)^2}\ \lambda_{2n} + \mathcal{J}_{s,\omega}^{[2n]}\,,
\end{aligned}
\tag{2.64}
$$

[10] We used (2.21) and (2.22).

$$0 = \left[z_{\Phi,0}^{(n)}\right]'' + \frac{1}{x-1}\ \left[z_{\Phi,0}^{(n)}\right]' + \frac{2(x-1)}{x(2-x)}\ \left[z_{K,0}^{(n)}\right]' + \frac{2}{3(x-1)^3}\ \zeta_{2n}' + \mathcal{J}_{s,\Phi}^{[2n]}\,, \tag{2.65}$$

$$0 = \left[z_{K,0}^{(n)}\right]'' + \frac{1}{x-1}\ \left[z_{K,0}^{(n)}\right]' + \frac{2}{3(x-1)^3}\ \kappa_{2n}' + \mathcal{J}_{s,K}^{[2n]}\,, \tag{2.66}$$

where the source terms $\{\mathcal{J}_{s,H}^{[2n]}, \mathcal{J}_{s,f}^{[2n]}, \mathcal{J}_{s,\omega}^{[2n]}, \mathcal{J}_{s,\Phi}^{[2n]}, \mathcal{J}_{s,K}^{[2n]}\}$ are functionals of the lower order solutions: $z_{H,0}^{(m)}$, $z_{f,0}^{(m)}$, $z_{\omega,0}^{(m)}$, $z_{\Phi,0}^{(m)}$, $z_{K,0}^{(m)}$, κ_{2m}, ξ_{2m}, η_{2m}, λ_{2m}, ζ_{2m} and $\beta_{1,m}$, with $m < n$. Explicit expressions for the source term functionals are available from the author upon request. Apart from $n = 0$ [34] and for $\{z_{K,0}^{(1)}, z_{H,0}^{(1)}, z_{\Phi,0}^{(1)}\}$ [29] these equations must be solved numerically. We use the same numerical approach as outlined in section 2.1.

2.2.1 Order $n = 0$

We find:

$$z_{H,0}^{(0)} = 2x - x^2\,, \qquad z_{f,0}^{(0)} = z_{\omega,0}^{(0)} = z_{\Phi,0}^{(0)} = z_{K,0}^{(0)} = 0\,, \tag{2.67}$$

$$\beta_{1,0} = 1\,. \tag{2.68}$$

2.2.2 Order $n = 1$

We find:

$$z_{K,0}^{(1)} = z_{H,0}^{(1)} = 0\,, \qquad z_{\Phi,0}^{(1)} = \frac{x(2-x)}{12(1-x)^2}\ \ln(2x - x^2)\,, \tag{2.69}$$

$$\beta_{1,1} = -\frac{2}{3}\,. \tag{2.70}$$

Near the boundary, $x \to 0_+$, we have

$$z_{f,0}^{(1)} = -\frac{1}{80}x + x^2\left(s_{f,1}^{4,0} - \frac{1}{60}\ln x\right) + \mathcal{O}(x^3 \ln x)\,, \tag{2.71}$$

$$z_{\omega,0}^{(1)} = -\frac{1}{45}x + x^{3/2}s_{\omega,1}^{3,0} + \mathcal{O}(x^2)\,. \tag{2.72}$$

Near the horizon, $y \to 0_+$, we have

$$z_{f,0}^{(1)} = s_{f,h}^{1,0} + \mathcal{O}(y^2)\,, \tag{2.73}$$

$$z_{\omega,0}^{(1)} = s_{\omega,h}^{1,0} + \mathcal{O}(y^2)\,. \tag{2.74}$$

Altogether at this order we have 4 integration constants

$$\{s_{f,1}^{4,0}\,, s_{\omega,1}^{3,0}\,, s_{f,h}^{1,0}\,, s_{\omega,h}^{1,0}\}\,, \tag{2.75}$$

which is precisely what is needed to specify a unique solution for $\{z_{f,0}^{(1)}, z_{\omega,0}^{(1)}\}$.

2.2.3 Order $n = 2$

Near the boundary, $x \to 0_+$, we have

$$z_{K,0}^{(2)} = x \left(s_{K,2}^{2,0} - \frac{1}{6} \ln x \right) + \mathcal{O}(x^{3/2}) \,, \tag{2.76}$$

$$z_{H,0}^{(2)} = x \ s_{H,2}^{2,0} + \mathcal{O}(x^2 \ln x) \,, \tag{2.77}$$

$$z_{f,0}^{(2)} = x \left(\frac{13}{360} - \frac{1}{480} \ln 2 + \frac{7}{40} s_{K,2}^{2,0} - \frac{1}{32} \ln x \right) - \frac{2}{15} x^{3/2} s_{\omega,1}^{3,0} + x^2 \Bigg(s_{f,2}^{4,0} + \Bigg(\frac{149}{4320} - \frac{11}{720} \ln 2 - \frac{1}{20} s_{K,2}^{2,0} - \frac{1}{60} \kappa_2^{2,0} - \frac{1}{72} \zeta_1^{2,0} - \frac{1}{2} s_{f,1}^{4,0} \Bigg) \ln x + \frac{1}{720} \ln^2 x \Bigg) + \mathcal{O}(x^{5/2}) \,, \tag{2.78}$$

$$z_{\omega,0}^{(2)} = x \left(-\frac{2}{15} s_{K,2}^{2,0} + \frac{4}{45} - \frac{1}{45} \ln 2 \right) + x^{3/2} \ s_{\omega,2}^{3,0} + \mathcal{O}(x^2 \ln x) \,, \tag{2.79}$$

$$z_{\Phi,0}^{(2)} = x \left(s_{\Phi,2}^{2,0} + \left(-\frac{2}{3} + \frac{1}{6} \ln 2 + s_{K,2}^{2,0} \right) \ln x \right) + \mathcal{O}(x^2 \ln x) \,. \tag{2.80}$$

Near the horizon, $y \to 0_+$, we have

$$z_{K,0}^{(2)} = s_{K,h}^{2,0} + \mathcal{O}(y^2) \,, \tag{2.81}$$

$$z_{H,0}^{(2)} = \Bigg(-2\beta_{1,2} + 320 s_{f,h}^{1,0} \ \eta_{1,h}^0 - 8 s_{K,h}^{2,0} - \frac{8}{9} - \frac{2}{3} \zeta_{1,h}^0 + \frac{1}{5} \lambda_{1,h}^0 - \frac{38}{3} \eta_{1,h}^0 + \frac{1}{2} (\lambda_{1,h}^0)^2 + 8 \xi_{2,h}^1 - 8 s_{\omega,h}^{1,0} + 24 s_{w,h}^{1,0} \ \lambda_{1,h}^0 + \frac{56}{3} s_{f,h}^{1,0} \Bigg) y^2 + \mathcal{O}(y^4) \,, \tag{2.82}$$

$$z_{f,0}^{(2)} = s_{f,h}^{2,0} + \mathcal{O}(y^2) \,, \tag{2.83}$$

$$z_{\omega,0}^{(2)} = s_{\omega,h}^{2,0} + \mathcal{O}(y^2) \,, \tag{2.84}$$

$$z_{\Phi,0}^{(2)} = s_{\Phi,h}^{2,0} + \mathcal{O}(y^2) \,. \tag{2.85}$$

Altogether at this order we have 10 integration constants

$$\{ s_{K,2}^{2,0} \,, s_{H,2}^{2,0} \,, s_{f,2}^{4,0} \,, s_{\omega,2}^{3,0} \,, s_{\Phi,2}^{2,0} \,, s_{K,h}^{2,0} \,, \beta_{1,2} \,, s_{f,h}^{2,0} \,, s_{\omega,h}^{2,0} \,, s_{\Phi,h}^{2,0} \} \,, \tag{2.86}$$

which is precisely what is needed to specify a unique solution for $\{z_{K,0}^{(2)}$, $z_{H,0}^{(2)}$, $z_{f,0}^{(2)}$, $z_{\omega,0}^{(2)}$, $z_{\Phi,0}^{(2)}\}$.

2.2.4 Order $n = 3$

Near the boundary, $x \to 0_+$, we have

$$z_{K,0}^{(3)} = x\left(s_{K,3}^{2,0} + \left(\frac{17}{24} - \frac{1}{6}\ln 2 - s_{K,2}^{2,0}\right)\ln x\right) + \mathcal{O}(x^{3/2})\,, \tag{2.87}$$

$$z_{H,0}^{(3)} = x\ s_{H,3}^{2,0} + \mathcal{O}(x^2 \ln^2 x)\,, \tag{2.88}$$

$$\begin{aligned} z_{f,0}^{(3)} = x\Bigg(&-\frac{3}{80}\beta_{1,2} + \frac{11}{480}\ln 2 - \frac{1}{960}\ln^2 2 - \frac{1}{12}s_{K,2}^{2,0} - \frac{1}{160}s_{H,2}^{2,0} + \frac{7}{40}s_{K,3}^{2,0} + \frac{1}{40}s_{\Phi,2}^{2,0} \\ &- \frac{11}{180} + \frac{7}{80}\ln 2\ s_{K,2}^{2,0} + \left(\frac{83}{576} - \frac{1}{24}\ln 2 - \frac{1}{16}s_{K,2}^{2,0}\right)\ln x - \frac{1}{64}\ln^2 x\Bigg) \\ &+ x^{3/2}\left(\frac{1}{9}s_{\omega,1}^{3,0} - \frac{1}{15}s_{\omega,1}^{3,0}\ \ln 2 - \frac{2}{15}s_{\omega,2}^{3,0} - \frac{1}{15}s_{\omega,1}^{3,0}\ \ln x\right) + x^2\Bigg(s_{f,3}^{4,0} + \Bigg(-\frac{7}{360}\ln 2\ \zeta_1^{2,0} \\ &- \frac{7}{120}\beta_{1,2} + \frac{7}{24}s_{f,1}^{4,0} + \frac{67}{360}s_{K,2}^{2,0} + \frac{239}{2160}\kappa_2^{2,0} + \frac{301}{4320}\zeta_1^{2,0} - \frac{1}{120}s_{H,2}^{2,0} + \frac{29}{960}\ln 2 - \frac{1}{2}s_{f,2}^{4,0} \\ &- \frac{1}{20}s_{K,3}^{2,0} - \frac{1}{72}\zeta_2^{2,0} - \frac{1}{24}s_{\Phi,2}^{2,0} - \frac{1}{60}\kappa_3^{2,0} - \frac{1}{12}\zeta_1^{2,0}s_{K,2}^{2,0} - \frac{1}{45}\kappa_2^{2,0}\ln 2 - \frac{1}{10}\kappa_2^{2,0}s_{K,2}^{2,0} \\ &- \frac{1}{4}\ln 2\ s_{f,1}^{4,0} - \frac{1}{40}\ln 2\ s_{K,2}^{2,0} - \frac{11}{1440}\ln^2 2 - \frac{6431}{103680}\Bigg)\ln x + \Bigg(\frac{1}{480}\ln 2 + \frac{1}{120}s_{K,2}^{2,0} \\ &+ \frac{1}{720}\zeta_1^{2,0} + \frac{1}{360}\kappa_2^{2,0} - \frac{461}{17280}\Bigg)\ln^2 x\Bigg) + \mathcal{O}(x^{5/2}\ln x)\,, \end{aligned} \tag{2.89}$$

$$\begin{aligned} z_{\omega,0}^{(3)} = x\Bigg(&-\frac{1}{15}\beta_{1,2} - \frac{89}{405} + \frac{2}{9}s_{K,2}^{2,0} - \frac{1}{90}s_{H,2}^{2,0} - \frac{2}{15}s_{K,3}^{2,0} - \frac{1}{15}s_{\Phi,2}^{2,0} + \frac{47}{540}\ln 2 - \frac{1}{90}\ln^2 2 \\ &- \frac{1}{15}\ln 2\ s_{K,2}^{2,0}\Bigg) + x^{3/2}\ s_{\omega,3}^{3,0} + \mathcal{O}(x^2\ln x)\,, \end{aligned} \tag{2.90}$$

$$\begin{aligned} z_{\Phi,0}^{(3)} = x\Bigg(&s_{\Phi,3}^{2,0} + \Bigg(\frac{1}{2}\beta_{1,2} - \frac{5}{3}s_{K,2}^{2,0} + \frac{1}{12}s_{H,2}^{2,0} + s_{K,3}^{2,0} + \frac{1}{2}s_{\Phi,2}^{2,0} + \frac{1}{12}\ln^2 2 - \frac{47}{72}\ln 2 \\ &+ \frac{1}{2}\ln 2\ s_{K,2}^{2,0} + \frac{89}{54}\Bigg)\ln x\Bigg) + \mathcal{O}(x^2\ln^2 x)\,. \end{aligned} \tag{2.91}$$

Near the horizon, $y \to 0_+$, we have

$$z_{K,0}^{(3)} = s_{K,h}^{3,0} + \mathcal{O}(y^2)\,, \tag{2.92}$$

$$z_{H,0}^{(3)} = \left(\frac{8}{3}\beta_{1,2} - 2\beta_{1,3} + \cdots\right)y^2 + \mathcal{O}(y^4)\,, \tag{2.93}$$

where $\cdots$ denote dependence on lower order coefficients, except for $\{\beta_{1,2}, \beta_{1,3}\}$ — the expression is too long to be presented here,

$$z^{(3)}_{f,0} = s^{3,0}_{f,h} + \mathcal{O}(y^2)\,, \tag{2.94}$$

$$z^{(3)}_{\omega,0} = s^{3,0}_{\omega,h} + \mathcal{O}(y^2)\,, \tag{2.95}$$

$$z^{(3)}_{\Phi,0} = s^{3,0}_{\Phi,h} + \mathcal{O}(y^2)\,. \tag{2.96}$$

Altogether at this order we have 10 integration constants

$$\{s^{2,0}_{K,3}\,, s^{2,0}_{H,3}\,, s^{4,0}_{f,3}\,, s^{3,0}_{\omega,3}\,, s^{2,0}_{\Phi,3}\,, s^{3,0}_{K,h}\,, \beta_{1,3}\,, s^{3,0}_{f,h}\,, s^{3,0}_{\omega,h}\,, s^{3,0}_{\Phi,h}\}\,, \tag{2.97}$$

which is precisely what is needed to specify a unique solution for $\{z^{(3)}_{K,0}$, $z^{(3)}_{H,0}$, $z^{(3)}_{f,0}$, $z^{(3)}_{\omega,0}$, $z^{(3)}_{\Phi,0}\}$.

2.2.5 Order $n = 4$

Near the boundary, $x \to 0_+$, we have

$$\begin{aligned} z^{(4)}_{K,0} = x\Bigg(s^{2,0}_{K,4} + \bigg(-\frac{1}{2}\beta_{1,2} - \frac{1}{12}\ln^2 2 + \frac{7}{4}s^{2,0}_{K,2} - \frac{1}{12}s^{2,0}_{H,2} - s^{2,0}_{K,3} - \frac{1}{2}s^{2,0}_{\Phi,2} - \frac{265}{144} \\ - \frac{1}{2}\ln 2\ s^{2,0}_{K,2} + \frac{17}{24}\ln 2\bigg)\ln x + \frac{1}{48}\ln^2 x\Bigg) + \mathcal{O}(x^{3/2}\ln x)\,, \end{aligned} \tag{2.98}$$

$$z^{(4)}_{H,0} = x\ s^{2,0}_{H,4} + \mathcal{O}(x^2 \ln^3 x)\,. \tag{2.99}$$

Near the horizon, $y \to 0_+$, we have

$$z^{(4)}_{K,0} = s^{4,0}_{K,h} + \mathcal{O}(y^2)\,, \tag{2.100}$$

$$\begin{aligned} z^{(4)}_{H,0} = \Bigg(\frac{8}{3}\beta_{1,3} - 2\beta_{1,4} + \bigg(-\frac{14}{3} - 48 s^{1,0}_{\omega,h}\ \lambda^{h,0}_1 - \frac{28}{3}\eta^{h,0}_1 - 640 s^{1,0}_{f,h}\ \eta^{h,0}_1 - \frac{2}{15}\lambda^{h,0}_1 \\ + 160(\eta^{h,0}_1)^2 - \frac{4}{3}\zeta^{h,0}_1 - \frac{112}{3}s^{1,0}_{f,h} + 16 s^{1,0}_{\omega,h} + 16 s^{2,0}_{K,h} + 32\xi^{h,1}_2 + \frac{13}{5}(\lambda^{h,0}_1)^2 \bigg)\beta_{1,2} \\ - 5\beta^2_{1,2} + \cdots \Bigg) y^2 + \mathcal{O}(y^4)\,, \end{aligned} \tag{2.101}$$

where $\cdots$ denote dependence on lower order coefficients, except for $\{\beta_{1,2}, \beta_{1,3}, \beta_{1,4}\}$ — the expression is too long to be presented here.

Altogether at this order we have 4 integration constants

$$\{s^{2,0}_{K,4}\,, s^{2,0}_{H,4}\,, s^{4,0}_{K,h}\,, \beta_{1,4}\}\,, \tag{2.102}$$

which is precisely what is needed to specify a unique solution for $\{z^{(4)}_{K,0}, z^{(4)}_{H,0}\}$.

n	1	2	3	4
$s_{K,n}^{2,0}$		0.07891997	-0.53177623	1.48077259
$s_{H,n}^{2,0}$		-0.76150758	2.66710077	-5.15078326
$s_{f,n}^{4,0}$	-0.01641302	0.02836594	-0.00340767	
$s_{\omega,n}^{3,0}$	0.04361787	-0.05326321	-0.05682864	
$s_{\Phi,n}^{2,0}$		-0.03656488	-0.55146784	
$s_{K,h}^{n,0}$		0.14236466	-0.61460152	1.49213973
$\beta_{1,n}$		0.33333333	0.340767665	-0.81625254
$s_{f,h}^{n,0}$	-0.00362335	0.038494905	-0.14888558	
$s_{\omega,h}^{n,0}$	-0.00413161	0.01711156	-0.04003014	
$s_{\Phi,h}^{n,0}$		0.33416570	-0.74238359	

Table 2: Coefficients of the normalizable modes of the sound quasinormal modes to order $\mathcal{O}(\mathfrak{q}^0)$. See (2.75), (2.86), (2.97) and (2.102).

2.2.6 Integration constants for the sound quasinormal modes at $\mathcal{O}(\mathfrak{q}^0)$

Here we tabulate (see table 2) the integration constants for the normalizable modes of $\{z_{K,0}^{(n)},\, z_{H,0}^{(n)},\, z_{f,0}^{(n)},\, z_{\omega,0}^{(n)},\, z_{\Phi,0}^{(n)}\}$. with $n = \{1, 2, 3, 4\}$ obtained from solving the corresponding boundary value problems.

2.3 Bulk viscosity of the cascading plasma to order $\mathcal{O}\left(\frac{P^8}{K_\star^4}\right)$

Equations of motion for the sound waves in the cascading plasma for generic $\frac{P^2}{K_\star}$ were derived in [29]. Previously, they have been discussed (solved) only to order $\mathcal{O}\left(\frac{P^2}{K_\star}\right)$, [29]. Here, we extend the analysis to $n = 4$ in (2.13) at order $\mathcal{O}(\mathfrak{q}^1)$.

Equations of motion for $\{z_{H,1}^{(n)}, z_{f,1}^{(n)}, z_{\omega,1}^{(n)}, z_{\Phi,1}^{(n)}, z_{K,1}^{(n)}\}$ take form[11]

$$
\begin{split}
0=&\left[z_{H,1}^{(n)}\right]''-\frac{3x^2-6x+2}{(x^2-2x+2)(x-1)}\left[z_{H,1}^{(n)}\right]'+\frac{4}{x^2-2x+2}z_{H,1}^{(n)}+\frac{32}{x^2-2x+2}z_{K,1}^{(n)}\\
&-\frac{2x^2(2-x)^2}{3^{1/2}(x-1)(x^2-2x+2)^2}\left[z_{H,0}^{(n)}\right]'+\frac{4x(x-2)}{3^{1/2}(x^2-2x+2)^2}z_{H,0}^{(n)}\\
&+\frac{64(2x^2-4x+3)}{3^{1/2}(x^2-2x+2)^2}z_{K,0}^{(n)}+\frac{16(x^2-2x+4)(2-x)^2x^2}{3^{1/2}(x-1)(x^2-2x+2)^2}\xi_{2n}'+\frac{16(x-2)x}{3^{3/2}(x-1)}\kappa_{2n}'\\
&-\frac{8\beta_{2,n}}{3^{1/2}(x^2-2x+2)}-\frac{8(x^2-2x+4)\beta_{1,n}}{3^{1/2}(x^2-2x+2)^2}+\mathcal{J}_{a,H}^{[2n]}\,,
\end{split}
\tag{2.103}
$$

$$
\begin{split}
0=&\left[z_{f,1}^{(n)}\right]''+\frac{1}{x-1}\left[z_{f,1}^{(n)}\right]'-\frac{8}{x^2(2-x)^2}z_{f,1}^{(n)}-\frac{3(x-1)}{10x(x-2)}\left[z_{K,1}^{(n)}\right]'\\
&+\frac{2}{x^2(2-x)^2}z_{K,1}^{(n)}-\frac{2}{3^{1/2}(x-1)}\left[z_{f,0}^{(n)}\right]'+\frac{3^{1/2}}{10x(x-2)}z_{K,0}^{(n)}\\
&+\frac{2}{3^{1/2}x(1-x)^2(x-2)}\kappa_{2n}+\mathcal{J}_{a,f}^{[2n]}\,,
\end{split}
\tag{2.104}
$$

$$
\begin{split}
0=&\left[z_{\omega,1}^{(n)}\right]''+\frac{1}{x-1}\left[z_{\omega,1}^{(n)}\right]'-\frac{3}{x^2(2-x)^2}z_{w,1}^{(n)}-\frac{(x-1)}{5x(x-2)}\left[z_{K,1}^{(n)}\right]'\\
&-\frac{2}{3^{1/2}(x-1)}\left[z_{\omega,0}^{(n)}\right]'+\frac{3^{1/2}}{15x(x-2)}z_{K,0}^{(n)}+\frac{3^{1/2}}{15(x-1)^3}\lambda_{2n}'\\
&+\frac{3^{1/2}}{10x(1-x)^2(x-2)}\lambda_{2n}+\mathcal{J}_{a,\omega}^{[2n]}\,,
\end{split}
\tag{2.105}
$$

$$
\begin{split}
0=&\left[z_{\Phi,1}^{(n)}\right]''+\frac{1}{x-1}\left[z_{\Phi,1}^{(n)}\right]'-\frac{2(x-1)}{x(x-2)}\left[z_{K,1}^{(n)}\right]'-\frac{2}{3^{1/2}(x-1)}\left[z_{\Phi,0}^{(n)}\right]'\\
&+\frac{2}{3^{1/2}x(x-2)}z_{K,0}^{(n)}-\frac{2}{3^{1/2}(x-1)^3}\zeta_{2n}'+\mathcal{J}_{a,\Phi}^{[2n]}\,,
\end{split}
\tag{2.106}
$$

$$
0=\left[z_{K,1}^{(n)}\right]''+\frac{1}{x-1}\left[z_{K,1}^{(n)}\right]'-\frac{2}{3^{1/2}(x-1)}\left[z_{K,0}^{(n)}\right]'-\frac{2}{3^{1/2}(x-1)^3}\kappa_{2n}'+\mathcal{J}_{a,K}^{[2n]}\,,\tag{2.107}
$$

where the source terms $\{\mathcal{J}_{a,H}^{[2n]}, \mathcal{J}_{a,f}^{[2n]}, \mathcal{J}_{a,\omega}^{[2n]}, \mathcal{J}_{a,\Phi}^{[2n]}, \mathcal{J}_{a,K}^{[2n]}\}$ are functionals of the lower order solutions: $z_{H,1}^{(m)}$, $z_{f,1}^{(m)}$, $z_{\omega,1}^{(m)}$, $z_{\Phi,1}^{(m)}$, $z_{K,1}^{(m)}$, $z_{H,0}^{(m)}$, $z_{f,0}^{(m)}$, $z_{\omega,0}^{(m)}$, $z_{\Phi,0}^{(m)}$, $z_{K,0}^{(m)}$, κ_{2m}, ξ_{2m}, η_{2m}, λ_{2m}, ζ_{2m} and $\{\beta_{1,m}, \beta_{2,m}\}$, with $m<n$. Explicit expressions for the source term functionals are available from the author upon request. Apart from $n=0$ [34] and for $\{z_{K,1}^{(1)}, z_{H,1}^{(1)}\}$ [29] these equations must be solved numerically. We use the same numerical approach as outlined in section 2.1.

[11]We used (2.21), (2.22) and (2.67), (2.68).

2.3.1 Order $n = 0$

We find:

$$z_{H,1}^{(0)} = z_{f,1}^{(0)} = z_{\omega,1}^{(0)} = z_{\Phi,1}^{(0)} = z_{K,1}^{(0)} = 0\,, \tag{2.108}$$

$$\beta_{2,0} = 1\,. \tag{2.109}$$

2.3.2 Order $n = 1$

We find:

$$z_{K,1}^{(1)} = z_{H,1}^{(1)} = 0\,, \tag{2.110}$$

$$\beta_{2,1} = \frac{2}{3}\,. \tag{2.111}$$

Near the boundary, $x \to 0_+$, we have

$$z_{f,1}^{(1)} = \frac{3^{1/2}}{80}x + x^2\left(a_{f,1}^{4,0} + \frac{7}{360}3^{1/2}\ln x\right) + \mathcal{O}(x^3\ln x)\,, \tag{2.112}$$

$$z_{\omega,1}^{(1)} = \frac{1}{45}3^{1/2}x + x^{3/2}\ a_{\omega,1}^{3,0} + \mathcal{O}(x^2)\,, \tag{2.113}$$

$$z_{\Phi,1}^{(1)} = x\left(a_{\Phi,1}^{2,0} - \frac{1}{6}3^{1/2}\ln x\right) + \mathcal{O}(x^2\ln x)\,. \tag{2.114}$$

Near the horizon, $y \to 0_+$, we have

$$z_{f,1}^{(1)} = a_{f,h}^{1,0} + \mathcal{O}(y^2)\,, \tag{2.115}$$

$$z_{\omega,1}^{(1)} = a_{\omega,h}^{1,0} + \mathcal{O}(y^2)\,, \tag{2.116}$$

$$z_{\Phi,1}^{(1)} = a_{\Phi,h}^{1,0} + \mathcal{O}(y^2)\,. \tag{2.117}$$

Altogether at this order we have 6 integration constants

$$\{a_{f,1}^{4,0}\,, a_{\omega,1}^{3,0}\,, a_{\Phi,1}^{2,0}\,, a_{f,h}^{1,0}\,, a_{\omega,h}^{1,0}\,, a_{\Phi,h}^{1,0}\}\,, \tag{2.118}$$

which is precisely what is needed to specify a unique solution for $\{z_{f,1}^{(1)}, z_{\omega,1}^{(1)}, z_{\Phi,1}^{(1)}\}$.

2.3.3 Order $n = 2$

Near the boundary, $x \to 0_+$, we have

$$z^{(2)}_{K,1} = x\left(a^{2,0}_{K,2} + \frac{1}{6}3^{1/2}\ln x\right) + \mathcal{O}(x^{3/2})\,, \tag{2.119}$$

$$z^{(2)}_{H,1} = x\ a^{2,0}_{H,2} + \mathcal{O}(x^2\ln x)\,, \tag{2.120}$$

$$\begin{aligned} z^{(2)}_{f,1} &= x\left(-\frac{1}{36}3^{1/2} + \frac{7}{40}a^{2,0}_{K,2} + \frac{1}{160}3^{1/2}\ln 2 + \frac{1}{40}a^{2,0}_{\Phi,1} + \frac{1}{32}3^{1/2}\ln x\right) - \frac{2}{15}x^{3/2}\ a^{3,0}_{\omega,1} \\ &+ x^2\Bigg(a^{4,0}_{f,2} + \Bigg(-\frac{1}{2}a^{4,0}_{f,1} - \frac{1}{20}a^{2,0}_{K,2} - \frac{1}{40}3^{1/2} - \frac{1}{24}a^{2,0}_{\Phi,1} + \frac{7}{720}3^{1/2}\ln 2 + \frac{1}{72}3^{1/2}\zeta^{2,0}_1 \\ &+ \frac{1}{60}3^{1/2}\kappa^{2,0}_2\Bigg)\ln x - \frac{1}{720}3^{1/2}\ln^2 x\Bigg) + \mathcal{O}(x^{5/2})\,, \end{aligned} \tag{2.121}$$

$$z^{(2)}_{\omega,1} = x\left(-\frac{2}{15}a^{2,0}_{K,2} - \frac{2}{27}3^{1/2} - \frac{1}{15}a^{2,0}_{\Phi,1} + \frac{1}{90}3^{1/2}\ln 2\right) + x^{3/2}\ a^{3,0}_{\omega,2} + \mathcal{O}(x^2\ln x)\,, \tag{2.122}$$

$$z^{(2)}_{\Phi,1} = x\left(a^{2,0}_{\Phi,2} + \left(a^{2,0}_{K,2} + \frac{5}{9}3^{1/2} + \frac{1}{2}a^{2,0}_{\Phi,1} - \frac{1}{12}3^{1/2}\ln 2\right)\ln x\right) + \mathcal{O}(x^2\ln x)\,. \tag{2.123}$$

Near the horizon, $y \to 0_+$, we have

$$z^{(2)}_{K,1} = a^{2,0}_{K,h} + \mathcal{O}(y^2)\,, \tag{2.124}$$

$$\begin{aligned} z^{(2)}_{H,1} = \Bigg(&-8a^{2,0}_{K,h} + \frac{56}{3}3^{1/2}s^{1,0}_{f,h} - 83^{1/2}s^{1,0}_{\omega,h} + \frac{8}{45}3^{1/2}\lambda^0_{1,h} - \frac{16}{3}3^{1/2}\eta^0_{1,h} + \frac{17}{54}3^{1/2} \\ &+ 320\ 3^{1/2}s^{1,0}_{f,h}\ \eta^0_{1,h} + 24\ 3^{1/2}s^{1,0}_{\omega,h}\ \lambda^0_{1,h} - \frac{16}{3}3^{1/2}\xi^1_{2,h} - 8\ 3^{1/2}s^{2,0}_{K,h} + \frac{56}{3}a^{1,0}_{f,h} - 8a^{1,0}_{\omega,h} \\ &- 2a^{1,0}_{\Phi,h} + \frac{4}{3}3^{1/2}\beta_{1,2} + \frac{2}{3}3^{1/2}\beta_{2,2} - \frac{160}{3}3^{1/2}(\eta^0_{1,h})^2 - \frac{8}{15}3^{1/2}(\lambda^0_{1,h})^2 + 320a^{1,0}_{f,h}\ \eta^0_{1,h} \\ &+ 24a^{1,0}_{\omega,h}\ \lambda^0_{1,h}\Bigg)y^2 + \mathcal{O}(y^4)\,, \end{aligned} \tag{2.125}$$

$$z^{(2)}_{f,1} = a^{2,0}_{f,h} + \mathcal{O}(y^2)\,, \tag{2.126}$$

$$z^{(2)}_{\omega,1} = a^{2,0}_{\omega,h} + \mathcal{O}(y^2)\,, \tag{2.127}$$

$$z^{(2)}_{\Phi,1} = a^{2,0}_{\Phi,h} + \mathcal{O}(y^2)\,. \tag{2.128}$$

Altogether at this order we have 10 integration constants

$$\{a^{2,0}_{K,2}\,, a^{2,0}_{H,2}\,, a^{4,0}_{f,2}\,, a^{3,0}_{\omega,2}\,, a^{2,0}_{\Phi,2}\,, a^{2,0}_{K,h}\,, \beta_{2,2}\,, a^{2,0}_{f,h}\,, a^{2,0}_{\omega,h}\,, a^{2,0}_{\Phi,h}\}\,, \tag{2.129}$$

which is precisely what is needed to specify a unique solution for $\{z^{(2)}_{K,1}$, $z^{(2)}_{H,1}$, $z^{(2)}_{f,1}$, $z^{(2)}_{\omega,1}$, $z^{(2)}_{\Phi,1}\}$.

2.3.4 Order $n = 3$

Near the boundary, $x \to 0_+$, we have

$$z^{(3)}_{K,1} = x\left(a^{2,0}_{K,3} + \left(-\frac{43}{72}3^{1/2} + \frac{1}{12}3^{1/2}\ln 2 - a^{2,0}_{K,2} - \frac{1}{2}a^{2,0}_{\Phi,1}\right)\ln x\right) + \mathcal{O}(x^{3/2})\,, \quad (2.130)$$

$$z^{(3)}_{H,1} = x\ a^{2,0}_{H,3} + \mathcal{O}(x^2\ln^2 x)\,, \quad (2.131)$$

$$\begin{aligned} z^{(3)}_{f,1} = x\Bigg(&\frac{49}{1080}3^{1/2} - \frac{1}{160}a^{2,0}_{H,2} + \frac{1}{160}3^{1/2}s^{2,0}_{H,2} - \frac{7}{360}3^{1/2}\ln 2 + \frac{7}{80}\ln 2\ a^{2,0}_{K,2} \\ &+ \frac{1}{320}3^{1/2}\ln^2 2 + \frac{1}{40}a^{2,0}_{\Phi,2} + \frac{1}{80}\ln 2\ a^{2,0}_{\Phi,1} + \frac{7}{40}a^{2,0}_{K,3} + \frac{3}{80}3^{1/2}\beta_{1,2} + \frac{1}{80}3^{1/2}\beta_{2,2} - \frac{1}{12}a^{2,0}_{K,2} \\ &- \frac{1}{240}a^{2,0}_{\Phi,1} + \left(-\frac{71}{576}3^{1/2} + \frac{1}{32}3^{1/2}\ln 2 - \frac{1}{16}a^{2,0}_{K,2} - \frac{1}{16}a^{2,0}_{\Phi,1}\right)\ln x + \frac{1}{64}3^{1/2}\ln^2 x\Bigg) \\ &+ x^{3/2}\left(\frac{1}{9}a^{3,0}_{\omega,1} - \frac{1}{15}a^{3,0}_{\omega,1}\ln 2 - \frac{2}{15}a^{3,0}_{\omega,2} - \frac{1}{15}a^{3,0}_{\omega,1}\ln x\right) + x^2\Bigg(a^{4,0}_{f,3} + \Bigg(\frac{3}{40}3^{1/2}\beta_{1,2} \\ &+ \frac{1}{360}3^{1/2}s^{2,0}_{K,2} + \frac{7}{360}3^{1/2}\beta_{2,2} + \frac{7}{720}3^{1/2}s^{2,0}_{H,2} - \frac{77}{1440}3^{1/2}\zeta^{2,0}_1 - \frac{1}{360}3^{1/2}\ln 2 \\ &- \frac{43}{432}3^{1/2}\kappa^{2,0}_2 + \frac{1}{120}3^{1/2}\kappa^{2,0}_2\ln 2 + \frac{1}{144}3^{1/2}\ln 2\zeta^{2,0}_1 + \frac{11753}{518400}3^{1/2} + \frac{1}{72}3^{1/2}\zeta^{2,0}_2 \\ &+ \frac{1}{60}3^{1/2}\kappa^{2,0}_3 + \frac{7}{1440}3^{1/2}\ln^2 2 + \frac{161}{1440}a^{2,0}_{\Phi,1} + \frac{67}{360}a^{2,0}_{K,2} - \frac{1}{120}a^{2,0}_{H,2} - \frac{3}{40}\zeta^{2,0}_1\ a^{2,0}_{\Phi,1} - \frac{1}{2}a^{4,0}_{f,2} \\ &- \frac{1}{24}a^{2,0}_{\Phi,2} - \frac{1}{12}\zeta^{2,0}_1\ a^{2,0}_{K,2} - \frac{1}{12}\kappa^{2,0}_2\ a^{2,0}_{\Phi,1} - \frac{1}{20}a^{2,0}_{K,3} - \frac{1}{48}\ln 2\ a^{2,0}_{\Phi,1} - \frac{1}{10}\kappa^{2,0}_2\ a^{2,0}_{K,2} \\ &- \frac{1}{40}\ln 2\ a^{2,0}_{K,2} - \frac{1}{4}\ln 2\ a^{4,0}_{f,1} + \frac{7}{24}a^{4,0}_{f,1}\Bigg)\ln x + \Bigg(-\frac{1}{720}3^{1/2}\ln 2 - \frac{1}{720}3^{1/2}\zeta^{2,0}_1 \\ &- \frac{1}{360}3^{1/2}\kappa^{2,0}_2 + \frac{19}{640}3^{1/2} + \frac{1}{240}a^{2,0}_{\Phi,1} + \frac{1}{120}a^{2,0}_{K,2}\Bigg)\ln^2 x\Bigg) + \mathcal{O}(x^{5/2}\ln x)\,, \end{aligned} \quad (2.132)$$

$$\begin{aligned} z^{(3)}_{\omega,1} = x\Bigg(&-\frac{1}{90}a^{2,0}_{H,2} - \frac{1}{15}a^{2,0}_{\Phi,2} - \frac{2}{15}a^{2,0}_{K,3} - \frac{1}{30}\ln 2\ a^{2,0}_{\Phi,1} + \frac{71}{405}3^{1/2} + \frac{2}{9}a^{2,0}_{K,2} + \frac{11}{90}a^{2,0}_{\Phi,1} \\ &+ \frac{1}{90}3^{1/2}s^{2,0}_{H,2} - \frac{1}{15}\ln 2\ a^{2,0}_{K,2} + \frac{1}{15}3^{1/2}\beta_{1,2} + \frac{1}{45}3^{1/2}\beta_{2,2} - \frac{8}{135}3^{1/2}\ln 2 + \frac{1}{180}3^{1/2}\ln^2 2\Bigg) \\ &+ x^{3/2}\ a^{3,0}_{\omega,3} + \mathcal{O}(x^2\ln x)\,, \end{aligned} \quad (2.133)$$

$$z^{(3)}_{\Phi,1} = x\Bigg(a^{2,0}_{\Phi,3} + \Bigg(-\frac{1}{12}3^{1/2}s^{2,0}_{H,2} + \frac{1}{12}a^{2,0}_{H,2} + \frac{1}{4}\ln 2\ a^{2,0}_{\Phi,1} + \frac{1}{2}\ln 2\ a^{2,0}_{K,2} + \frac{1}{2}\ a^{2,0}_{\Phi,2} + a^{2,0}_{K,3}$$
$$- \frac{71}{54}3^{1/2} - \frac{5}{3}a^{2,0}_{K,2} - \frac{11}{12}a^{2,0}_{\Phi,1} - \frac{1}{2}3^{1/2}\beta_{1,2} - \frac{1}{6}3^{1/2}\beta_{2,2} + \frac{4}{9}3^{1/2}\ln 2 - \frac{1}{24}3^{1/2}\ln^2 2\Bigg)\ln x\Bigg)$$
$$+ \mathcal{O}(x^2\ln^2 x)\,. \tag{2.134}$$

Near the horizon, $y \to 0_+$, we have

$$z^{(3)}_{K,1} = a^{3,0}_{K,h} + \mathcal{O}(y^2)\,, \tag{2.135}$$

$$z^{(3)}_{H,1} = \Bigg(-\frac{8}{9}3^{1/2}\beta_{2,2} + \frac{2}{3}3^{1/2}\beta_{2,3} + \cdots\Bigg)y^2 + \mathcal{O}(y^4)\,, \tag{2.136}$$

where $\cdots$ denote dependence on lower order coefficients, except for $\{\beta_{2,2}, \beta_{2,3}\}$ — the expression is too long to be presented here,

$$z^{(3)}_{f,1} = a^{3,0}_{f,h} + \mathcal{O}(y^2)\,, \tag{2.137}$$

$$z^{(3)}_{\omega,1} = a^{3,0}_{\omega,h} + \mathcal{O}(y^2)\,, \tag{2.138}$$

$$z^{(3)}_{\Phi,1} = a^{3,0}_{\Phi,h} + \mathcal{O}(y^2)\,. \tag{2.139}$$

Altogether at this order we have 10 integration constants

$$\{a^{2,0}_{K,3}\,, a^{2,0}_{H,3}\,, a^{4,0}_{f,3}\,, a^{3,0}_{\omega,3}\,, a^{2,0}_{\Phi,3}\,, a^{3,0}_{K,h}\,, \beta_{2,3}\,, a^{3,0}_{f,h}\,, a^{3,0}_{\omega,h}\,, a^{3,0}_{\Phi,h}\}\,, \tag{2.140}$$

which is precisely what is needed to specify a unique solution for $\{z^{(3)}_{K,1}$, $z^{(3)}_{H,1}$, $z^{(3)}_{f,1}$, $z^{(3)}_{\omega,1}$, $z^{(3)}_{\Phi,1}\}$.

2.3.5 Order $n = 4$

Near the boundary, $x \to 0_+$, we have

$$z^{(4)}_{K,1} = x\Bigg(a^{2,0}_{K,4} + \Bigg(-\frac{1}{12}a^{2,0}_{H,2} - \frac{1}{2}a^{2,0}_{\Phi,2} - a^{2,0}_{K,3} - \frac{35}{72}3^{1/2}\ln 2 + \frac{7}{4}a^{2,0}_{K,2} + a^{2,0}_{\Phi,1} + \frac{1}{6}3^{1/2}\beta_{2,2}$$
$$+ \frac{1}{24}3^{1/2}\ln^2 2 + \frac{71}{48}3^{1/2} + \frac{1}{12}3^{1/2}s^{2,0}_{H,2} - \frac{1}{2}\ln 2\ a^{2,0}_{K,2} - \frac{1}{4}\ln 2\ a^{2,0}_{\Phi,1} + \frac{1}{2}3^{1/2}\beta_{1,2}\Bigg)\ln x$$
$$- \frac{1}{48}3^{1/2}\ln^2 x\Bigg) + \mathcal{O}(x^{3/2}\ln x)\,, \tag{2.141}$$

$$z^{(4)}_{H,1} = x\ a^{2,0}_{H,4} + \mathcal{O}(x^2\ln^3 x)\,. \tag{2.142}$$

Near the horizon, $y \to 0_+$, we have

$$z_{K,1}^{(4)} = a_{K,h}^{4,0} + \mathcal{O}(y^2)\,, \tag{2.143}$$

$$\begin{split} z_{H,1}^{(4)} = &\left(\frac{2}{3}3^{1/2}\beta_{2,4} - \frac{8}{9}3^{1/2}\beta_{2,3} + \left(-\frac{16}{3}3^{1/2}s_{\omega,h}^{1,0} - \frac{13}{15}3^{1/2}(\lambda_{1,h}^0)^2 - \frac{160}{3}3^{1/2}(\eta_{1,h}^0)^2\right.\right.\\ &- \frac{32}{3}3^{1/2}\xi_{2,h}^1 - \frac{16}{3}3^{1/2}s_{K,h}^{2,0} + \frac{10}{3}3^{1/2}\beta_{1,2} + \frac{4}{9}3^{1/2}\zeta_{1,h}^0 + \frac{640}{3}3^{1/2}s_{f,h}^{1,0}\ \eta_{1,h}^0\\ &\left.\left.+ 16\ 3^{1/2}s_{\omega,h}^{1,0}\ \lambda_{1,h}^0 + \frac{112}{9}3^{1/2}s_{f,h}^{1,0} + \frac{28}{9}3^{1/2}\eta_{1,h}^0 + \frac{2}{45}3^{1/2}\lambda_{1,h}^0 + \frac{14}{9}3^{1/2}\right)\beta_{2,2} + \cdots\right)y^2\\ &+ \mathcal{O}(y^4)\,, \end{split} \tag{2.144}$$

where $\cdots$ denote dependence on lower order coefficients, except for $\{\beta_{2,2}, \beta_{2,3}, \beta_{2,4}\}$ — the expression is too long to be presented here.

Altogether at this order we have 4 integration constants

$$\{a_{K,4}^{2,0}\,, a_{H,4}^{2,0}\,, a_{K,h}^{4,0}\,, \beta_{2,4}\}\,, \tag{2.145}$$

which is precisely what is needed to specify a unique solution for $\{z_{K,1}^{(4)}, z_{H,1}^{(4)}\}$.

2.3.6 Integration constants for the sound quasinormal modes at $\mathcal{O}(\mathfrak{q}^1)$

Here we tabulate (see table 3) the integration constants for the normalizable modes of $\{z_{K,1}^{(n)}, z_{H,1}^{(n)}, z_{f,1}^{(n)}, z_{\omega,1}^{(n)}, z_{\Phi,1}^{(n)}\}$. with $n = \{1, 2, 3, 4\}$ obtained from solving the corresponding boundary value problems.

3 Challenges of computing transport coefficients to all orders in $\frac{P^2}{K_\star}$

In the previous section we detailed the computation of the speed of sound and the bulk viscosity of the cascading plasma, perturbatively in $\frac{P^2}{K_\star}$ (note that $\frac{P^2}{K_\star} \sim \left(\ln \frac{T}{\Lambda}\right)^{-1}$ for[12] $T \gg \Lambda$ (1.4).) The results of the analysis presented in section 4 indicate that although $\mathcal{O}\left(\frac{P^8}{K_\star^4}\right)$ perturbative expansion is in excellent agreement with the full (nonperturbative in $\frac{P^2}{K_\star}$ computation for c_s^2 in the high temperature regime, this expansion does not converge below $T \simeq (1 \cdots 1.5)\Lambda$, which is about twice as high as the temperature of

[12]For exact temperature dependence of $K_\star$ see [22].

n	1	2	3	4
$a_{K,n}^{2,0}$		0.01069638	0.23174494	-0.98113866
$a_{H,n}^{2,0}$		0.08509403	-0.78349123	3.54269325
$a_{f,n}^{4,0}$	0.03737595	-0.06797689	0.08231942	
$a_{\omega,n}^{3,0}$	-0.08394261	0.10167870	-0.05749855	
$a_{\Phi,n}^{2,0}$	-0.29631916	0.38704391	-0.14363558	
$a_{K,h}^{n,0}$		-0.18958164	0.69624351	-1.81005605
$\beta_{2,n}$		0.13225837	-1.69770959	2.26988336
$a_{f,h}^{n,0}$	0.00580530	-0.05087126	0.17559889	
$a_{\omega,h}^{n,0}$	0.00630707	-0.02271965	0.05689077	
$a_{\Phi,h}^{n,0}$	0.11330816	-0.38874716	0.95684007	

Table 3: Coefficients of the normalizable modes of the sound quasinormal modes to order $\mathcal{O}(\mathfrak{q}^1)$. See (2.118), (2.129), (2.140) and (2.145).

the deconfinement phase transition. Thus, using perturbative analysis only we can not compute the bulk viscosity of the cascading plasma at the transition point. In this section we would like to explain the difficulty in going beyond the perturbative analysis as the latter might effect the analysis of other quasinormal modes in the cascading plasma, specifically those that could be responsible for the chiral symmetry breaking transition [24].

To understand the problem, it is instructive to go back to the numerical computation of the cascading plasma equilibrium equation of state. This was solved both perturbatively and non-perturbatively in $\frac{P^2}{K_\star}$ in [22]. On the dual gravitational side this computation involves finding the black hole solution in asymptotic KT geometry, *i.e.,* determining the gravitational fields $\{h\,, f_2\,, f_3\,, K\,, g\}$ (2.3)-(2.7). Above gravitational fields have non-normalizable (in some cases singular) modes both near the horizon $x\to 1_-$ and near the boundary $x\to 0_+$. Thus numerical integration must be done on an open interval $x\in(0,1)$, *i.e.,* we need to provide the boundary conditions for the gravitational fields as the series expansion in x near the boundary and in $y=1-x$ near the horizon. These series expansions must be fairly precise since, for examples the coefficient of the normalizable mode for f_2 (dual to the vev of the dimension eight gauge invariant operator of the cascading plasma) enters at order x^2 near the boundary, which is subdominant to coefficients $x^{n/2}\ln^k(x)$ with $n=0,1,2,3$ of the general

boundary expansion [22]

$$f_2 = a_0 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{n,k}\ x^{n/2}\ \ln^k x\,. \tag{3.1}$$

What saves the day, and ultimately allows for the full non-perturbative computations, is the fact that at each fixed order n, the maximum power of $\ln x$ in (3.1) happens to be bounded, $k \le n$. Thus, the series expansions of the type (3.1) are just generalized Taylor series expansions, which can be easily determined to any given order in n — the total number of expansion coefficients at order n grows as $\mathcal{O}(n^2)$. The situation would have been completely different, had the summation of k extend to infinity. Here one would have to solve exactly for the series in $(P^2 \ln x)$ at each order in x. Given the complexity of the equations involved the latter appears to be impossible.

Unfortunately, precisely this problem occurs in computation of the quasinormal modes in the sound channel. Consider for example the gauge invariant fluctuation z_f (2.8). As for the gravitational field f_2, it depends on the vev of the dimension eight operator — so its exact boundary asymptotic can not be specified with an accuracy of less than $\mathcal{O}(x^2)$[13]. Collecting (2.71), (2.78), (2.89) we find

$$z_{f,0} = x\left(-\frac{1}{80}\frac{P^2}{K_\star}\ln^0 x + \frac{1}{720}\frac{P^4}{K_\star^2}\ln^1 x - \frac{1}{64}\frac{P^6}{K_\star^3}\ln^2 x + \cdots\right) + \mathcal{O}\left(\frac{P^{2k}}{K_\star^k}\ x^2 \ln^k x\right)\,, \tag{3.2}$$

where we explicitly indicated only the leading $\ln x$ dependence at each order $\frac{P^{2k}}{K_\star^k}$. We further verified that $z_{f,0}^{(4)} = \mathcal{O}\left(x\ \ln^3 x\right)$ as $x \to 0_+$, and that in fact all the perturbative expansions for $\{z_H, z_f, z_\omega, z_\Phi, z_K\}$ do not truncate in k — for example,

$$z_{f,0} = \sum_{n=2}^{\infty} \sum_{k=0}^{\infty} s_f^{n,k}\ x^{n/2}\ \ln^k x\,. \tag{3.3}$$

It would be interesting to develop computational techniques to deal with this difficulty. Notice that the high temperature perturbative expansion provides an effective cutoff on the power of $\ln x$ in the boundary asymptotics since each additional factor of $\ln x$ comes with a factor of $\frac{P^2}{K_\star}$.

[13]Of course, in order to get reliable numerical results boundary asymptotics must be more precise — in our high temperature analysis we used expansions to order $\mathcal{O}(x^{9/2})$, which is five more orders beyond the highest order at which the normalizable coefficients of the fluctuations enter.

4 Perturbative hydrodynamics of the cascading plasma

In this section we present results of the perturbative high temperature analysis of the speed of sound waves and the bulk viscosity in the cascading plasma. We begin with discussion of the consistency checks on our analysis. Next, we move towards discussion of comparison between exact speed of sound (as given by (2.60)) and its perturbative high temperature expansion. This will allow us to comment on the convergence properties of the high temperature expansion. Extending the numerical analysis of [22] we show that chirally symmetric deconfined phase of the cascading gauge theory plasma becomes perturbatively unstable below the critical temperature of the deconfinement transition $T_{unstable} = 0.8749(0) T_{critical}$. We comment on the possible source of the instability. Finally, we discuss the bulk viscosity bound of [31] for the cascading plasma.

4.1 Consistency of analysis

4.1.1 The first law of thermodynamics

Cascading gauge theory plasma has a single scale Λ. It only makes sense to discuss the thermodynamics/hydrodynamics of the theory, provided one keeps Λ fixed. As explained in [22], enforcing that Λ is temperature independent leads to the following condition

$$\frac{d}{dT}\left(K_\star - \frac{P^2}{2}\ln a_0^2\right) = 0\,, \qquad a_0 \equiv \lim_{x\to 0_+} f_2\,. \tag{4.1}$$

From (2.4) we find

$$\begin{split} a_0 =& \tilde{a}_0\biggl(1 - \frac{1}{6}\frac{P^2}{K_\star} + \left(\frac{1}{18}\ln 2 - \frac{1}{12}\right)\frac{P^4}{K_\star^2} - \left(\frac{1}{54}\ln^2 2 + \frac{49}{648} - \frac{17}{216}\ln 2\right)\frac{P^6}{K_\star^3} \\ &+ \left(-\frac{55}{64} + \frac{29}{243}\ln 2 + \frac{1}{162}\ln^3 2 - \frac{11}{216}\ln^2 2\right)\frac{P^8}{K_\star^4} + \mathcal{O}\left(\frac{P^{10}}{K_\star^5}\right)\biggr)\,. \end{split} \tag{4.2}$$

Next, we compute the temperature of the black hole (2.1) using (2.3)-(2.4), and an explicit expression for G_{xx} — given by eq. (2.6) of [22]. We further invert the temperature relation to obtain $\tilde{a}_0$, and ultimately from (4.2) the perturbative expression for

a_0:

$$
\begin{aligned}
a_0 =&\frac{T^2\pi^2 K_\star}{4}\Bigg(1+\frac{1}{2}\frac{P^2}{K_\star}-\left(-\frac{5}{12}+\frac{1}{6}\ln 2+15(\eta^0_{1,h})^2+6\xi^0_{2,h}+\frac{2}{5}(\lambda^0_{1,h})^2\right)\frac{P^4}{K_\star^2}\\
&+\bigg(\frac{95}{21}-\frac{25}{72}\ln 2-3\xi^0_{2,h}-\frac{15}{2}(\eta^0_{1,h})^2-\frac{1}{5}(\lambda^0_{1,h})^2-6\xi^0_{3,h}+\frac{1}{18}\ln^2 2-\frac{4}{5}\lambda^0_{1,h}\ \lambda^0_{2,h}\\
&+\frac{4}{5}\eta^0_{1,h}\ (\lambda^0_{1,h})^2-30\eta^0_{1,h}\ \eta^0_{2,h}-40(\eta^0_{1,h})^3+\frac{4}{25}(\lambda^0_{1,h})^3+30\xi^0_{2,h}\ \eta^0_{1,h}\bigg)\frac{P^6}{K_\star^3}\\
&+\mathcal{O}\left(\frac{P^8}{K_\star^4}\right)\Bigg)\,.
\end{aligned}
\tag{4.3}
$$

Using (4.3) we find from (4.1)

$$
\begin{aligned}
\frac{dK_\star}{d\ln T}\bigg|_{d\Lambda=0} \equiv&\mathcal{A}^a_2\ P^2+\mathcal{A}^a_4\ \frac{P^4}{K_\star}+\mathcal{A}^a_6\ \frac{P^6}{K_\star^2}+\mathcal{A}^a_8\ \frac{P^8}{K_\star^3}+\mathcal{O}\left(\frac{P^{10}}{K_\star^4}\right)\\
=&2P^2+\frac{2P^4}{K_\star}+\frac{P^6}{K_\star^2}+\left(-\frac{7}{6}+\frac{2}{3}\ln 2+60(\eta^0_{1,h})^2+24\xi^0_{2,h}+\frac{8}{5}(\lambda^0_{1,h})^2\right)\frac{P^8}{K_\star^3}\\
&+\mathcal{O}\left(\frac{P^{10}}{K_\star^4}\right)\,.
\end{aligned}
\tag{4.4}
$$

Given (2.57), (2.58) and the expression for the entropy density s of the black hole (2.1), the first law of thermodynamics

$$
d\mathcal{P}=sdT\,,
$$

n	1	2	3	4
$\left(1-\frac{\mathcal{A}^b_{2n}}{\mathcal{A}^a_{2n}}\right)$	0	8.2×10^{-9}	-3.5×10^{-7}	-6.1×10^{-7}

Table 4: Comparison between $\mathcal{A}^a_{2n}$ and $\mathcal{A}^b_{2n}$ of (4.4) and (4.5).

leads to an alternative expression for $\frac{dK_\star}{d\ln T}$:

$$
\begin{split}
\frac{dK_\star}{d\ln T}\bigg|_{d\mathcal{P}-sdT=0} \equiv & \ \mathcal{A}^b_2\ P^2+\mathcal{A}^b_4\ \frac{P^4}{K_\star}+\mathcal{A}^b_6\ \frac{P^6}{K_\star^2}+\mathcal{A}^b_8\ \frac{P^8}{K_\star^3}+\mathcal{O}\left(\frac{P^{10}}{K_\star^4}\right)\\
=&2P^2+\left(-\frac43+2\zeta_1^{2,0}+4\kappa_2^{2,0}+\ln 2\right)\frac{P^4}{K_\star}+\biggl(4\kappa_3^{2,0}-\frac83\kappa_2^{2,0}-\frac85(\lambda^0_{1,h})^2-\frac43\zeta_1^{2,0}+\frac{79}{18}\\
&-60(\eta^0_{1,h})^2+2\zeta_2^{2,0}+2\kappa_2^{2,0}\ln 2+\frac12\ln^2 2-3\ln 2+\ln 2\ \zeta_1^{2,0}-24\xi^0_{2,h}\biggr)\frac{P^6}{K_\star^2}\\
&+\biggl(\frac{55}{9}\ln 2+\frac{91}{18}\zeta_1^{2,0}+\frac{91}{9}\kappa_2^{2,0}-\frac73\zeta_2^{2,0}-\frac{14}{3}\kappa_3^{2,0}+2\zeta_3^{2,0}+4\kappa_4^{2,0}+16\xi^0_{2,h}-\frac72\ln 2\ \zeta_1^{2,0}\\
&-7\kappa_2^{2,0}\ln 2-\frac94\ln^2 2+\ln 2\ \zeta_2^{2,0}+\frac12\ln^2 2\ \zeta_1^{2,0}+2\ln 2\ \kappa_3^{2,0}+\ln^2 2\ \kappa_2^{2,0}+\frac14\ln^3 2\\
&+40(\eta^0_{1,h})^2+\frac{16}{15}(\lambda^0_{1,h})^2-36\xi^0_{3,h}-180\eta^0_{1,h}\ \eta^0_{2,h}+\frac{24}{5}\eta^0_{1,h}\ (\lambda^0_{1,h})^2-\frac{24}{5}\lambda^0_{1,h}\ \lambda^0_{2,h}\\
&-240(\eta^0_{1,h})^3+\frac{24}{25}(\lambda^0_{1,h})^3+180\xi^0_{2,h}\ \eta^0_{1,h}-30\ln 2\ (\eta^0_{1,h})^2-12\ln 2\ \xi^0_{2,h}-\frac45\ln 2\ (\lambda^0_{1,h})^2\\
&-\frac{607}{108}-120(\eta^0_{1,h})^2\ \kappa_2^{2,0}-60(\eta^0_{1,h})^2\ \zeta_1^{2,0}-48\xi^0_{2,h}\ \kappa_2^{2,0}-24\xi^0_{2,h}\ \zeta_1^{2,0}-\frac{16}{5}(\lambda^0_{1,h})^2\ \kappa_2^{2,0}\\
&-\frac85(\lambda^0_{1,h})^2\ \zeta_1^{2,0}\biggr)\frac{P^8}{K_\star^3}+\mathcal{O}\left(\frac{P^{10}}{K_\star^4}\right)\,.
\end{split}
\tag{4.5}
$$

Comparison between $\mathcal{A}^a_{2n}$ and $\mathcal{A}^b_{2n}$ provides a highly nontrivial test on the consistency of the analysis. Using the data from the table 1, the results of such comparison are presented in table 4.

4.1.2 c_s^2 from the equilibrium thermodynamics and from the hydrodynamics

Our second consistency test compares the predictions of the equilibrium thermodynamics for $\beta_{1,n}=\beta_{1,n}^{thermo}$ from (2.61) with the direct computation of $\beta_{1,n}=\beta_{1,n}^{sound}$ (see (2.14) for the parametrization and table 2 for the results) . Using the data from the table 2, the results of such comparison are presented in table 5.

n	1	2	3	4
$\left(1-\frac{\beta_{1,n}^{thermo}}{\beta_{1,n}^{sound}}\right)$	0	-1.8×10^{-8}	4.0×10^{-7}	5.5×10^{-7}

Table 5: Comparison of coefficients $\beta_{1,n}$ in the sound wave dispersion relation (2.14) with the predicted values from the equilibrium thermodynamics, (2.61).

4.2 Speed of sound and perturbative instability of deconfined chirally symmetric phase of the cascading plasma at low temperatures

The speed of the sound waves can be computed from the dispersion relation of the quasinormal modes in the sound channel. In the high temperature expansion it is given by (see (2.14))

$$
\begin{split}
c_s^2\Big|_{high-T} =&\frac13\biggl\{1-\frac43\frac{P^2}{K_\star}+\left(2\beta_{1,2}+\frac49\right)\frac{P^4}{K_\star^2}+\left(2\beta_{1,3}-\frac43\beta_{1,2}\right)\frac{P^6}{K_\star^3}\\
&+\left(2\beta_{1,4}-\frac43\beta_{1,3}+(\beta_{1,2})^2\right)\frac{P^8}{K_\star^4}+\mathcal{O}\left(\frac{P^{10}}{K_\star^5}\right)\biggr\}\,,
\end{split}
\tag{4.6}
$$

where the coefficients $\beta_{1,n}$ are given in table 2. Alternatively, it can be evaluated from the equilibrium thermodynamics for any temperature (2.60),

$$
c_s^2\Big|_{thermo}=\frac{\partial\mathcal{P}}{\partial\mathcal{E}}=\frac13\ \frac{7-12\hat{a}_{2,0}-6P^2\ \frac{d\hat{a}_{2,0}}{dK_\star}}{7+4\hat{a}_{2,0}+2P^2\ \frac{d\hat{a}_{2,0}}{dK_\star}}\,.
\tag{4.7}
$$

From data in table 5, we see that there is an excellent agreement between (4.6) and the high temperature expansion of (2.60). Figure 1 represents comparison between (2.60) and (4.6) over a wide range of temperatures, *i.e.,* not necessarily when $\frac{P^2}{K_\star}\ll 1$. The blue dots represent the speed of sound computed from (2.60), slightly improving the analysis in [22]. The solid lines represent successive high temperature approximations to the speed of sound wave in the cascading plasma (4.6) to orders $\mathcal{O}\left(\frac{P^2}{K_\star}\right)$ (black), $\mathcal{O}\left(\frac{P^4}{K_\star^2}\right)$ (purple), $\mathcal{O}\left(\frac{P^6}{K_\star^3}\right)$ (green) and $\mathcal{O}\left(\frac{P^8}{K_\star^4}\right)$ (blue). It is convenient to plot the data with respect to $k_s\equiv\frac{K_\star}{P^2}-\frac12\ln 2$, rather than with respect to $\frac{T}{\Lambda}$. The vertical red line represents the deconfinement temperature of the cascading gauge theory plasma. Notice that the second $\mathcal{O}\left(\frac{P^4}{K_\star^2}\right)$ and the higher orders of the high temperature expansions become indistinguishable with the exact numerical data (blue dots) for $k_s\gtrsim 3$. Using

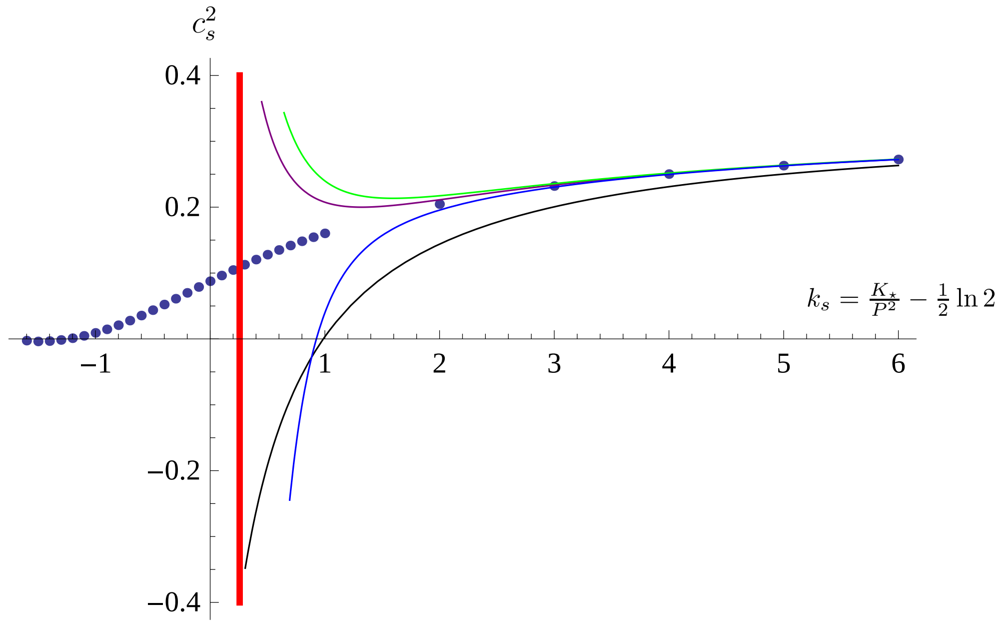

Figure 1: (color online) Speed of sound (blue dots) vs. its successive high temperature approximations. The vertical red line represents the temperature of the deconfinement phase transition in the cascading plasma.

results of [22]

$$\begin{aligned} k_s = 3 \qquad &\Longleftrightarrow \qquad \frac{T}{\Lambda} \approx 1.43 \qquad \Longleftrightarrow \qquad \frac{T}{T_{critical}} \approx 2.34\,, \\ k_s = 2 \qquad &\Longleftrightarrow \qquad \frac{T}{\Lambda} \approx 1.00 \qquad \Longleftrightarrow \qquad \frac{T}{T_{critical}} \approx 1.63\,, \end{aligned} \tag{4.8}$$

which suggests that the high temperature expansion converges for temperatures above $T \gtrsim (1 \cdots 1.5)\Lambda$.

Numerical analysis of the equilibrium thermodynamics in [22] were done to temperatures only slightly below the deconfinement temperature. Here, we extend the computations to lower temperatures. Notice from figure 1 that the speed of sound squared c_s^2 appears to cross zero (and turns negative) for $k_s < -1$. A more detailed analysis presented in figure 2 show that this is indeed so. A speed of sound vanishes at

$$c_s^2(k_{unstable}) = 0 \qquad \Longrightarrow \qquad k_{unstable} = -1.230(3)\,. \tag{4.9}$$

While $c_s^2 \sim (k_s - k_{unstable})$ in the vicinity of the instability, we find

$$c_s^2 \sim \left(1 - \frac{T_{unstable}}{T}\right)^{1/2}\,, \tag{4.10}$$

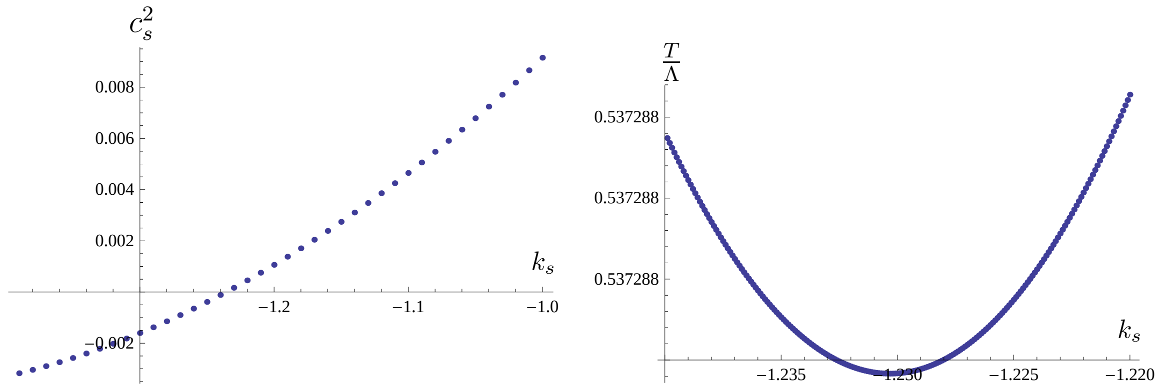

Figure 2: Speed of sound in the vicinity of the perturbative instability of the cascading plasma.

where

$$\frac{T_{unstable}}{\Lambda} = 0.53728(8)\,, \qquad \text{or} \qquad T_{unstable} = 0.8749(0) T_{critical}\,. \tag{4.11}$$

In fact, the near-unstable thermodynamics of the cascading theory is rather interesting. In figure 3 we show the free energy density as a function of k_s (left plot) and as a function of $\frac{T}{\Lambda}$ (right plot). Notice that there are two phases separated by a continuous phase transition at $k_s = k_{unstable}$. The phase with $k_s > k_{unstable}$ (which is continuously connected to a high temperature deconfined chirally symmetric phase of the cascading plasma) has a lower free energy compare to a phase with $k_s < k_{unstable}$ — the latter two phases are degenerate in temperature (see the right plot on figure 2) with the limiting temperature being reached precisely at $k_s = k_{unstable}$. It is clear from the right plot of figure 3 that[14]

$$\frac{\partial \mathcal{F}}{\partial T}\bigg|_{T\to T_{unstable}}$$

is finite. Thus, the only way the speed of sound can vanish at $T = T_{unstable}$ is if the rate of change of the energy density diverges at the unstable point. The plots in figure 4 show that this is indeed the case.

Vanishing of the speed of sound as in (4.10) implies that the specific heat c_V of the cascading plasma diverges near the unstable point with the critical exponent $\alpha = 0.5$[15]:

$$c_V = \frac{s}{c_s^2} \propto \left|1 - \frac{T_{unstable}}{T}\right|^{-1/2}\,. \tag{4.12}$$

[14]We did further, more detailed, numerical analysis to confirm this.

[15]This coincides with the mean-field critical exponent α at the tricritical point [35].

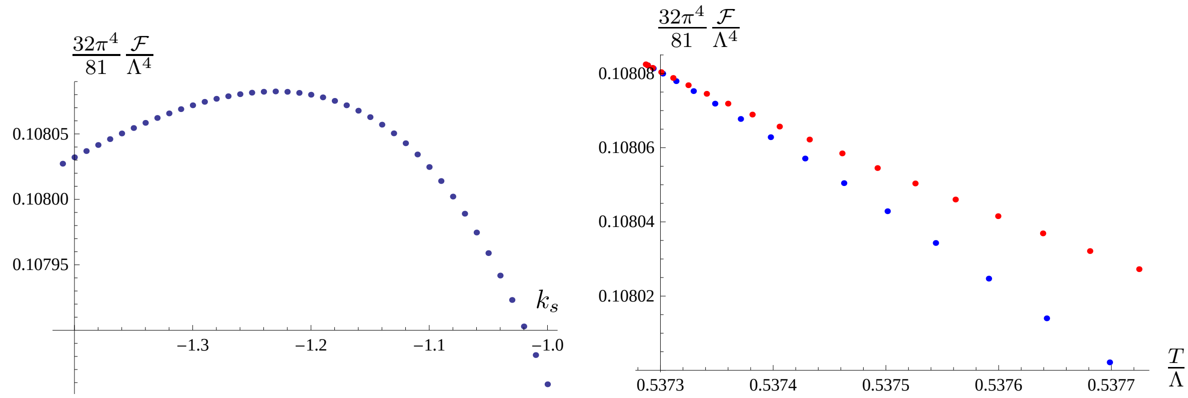

Figure 3: (color online) The free energy density in the vicinity of the perturbative instability of the cascading plasma. The red dots on the right plot correspond to $k_s \le k_{unstable}$ and the blue dots correspond to $k_s \ge k_{unstable}$.

Exactly the same critical behavior was found in $\mathcal{N} = 2^*$ plasma with mass deformation parameters $m_f < m_b$ [19, 31].

Whenever $c_s^2 < 0$, the thermodynamic system is unstable with respect to density fluctuations. It is interesting to understand the source of this instability. Recall that the zero temperature the vacuum of the cascading gauge theory spontaneously breaks chiral symmetry. Thus, it is conceivable that the perturbative instability observed in the equilibrium thermodynamics of the deconfined chirally symmetric phase of the cascading plasma is associated with the formation of chiral condensates. We comment more on this in the conclusion and for further analysis refer to future work [24].

4.3 Bulk viscosity bound in the cascading plasma

The primary goal of hydrodynamic analysis of the cascading plasma presented here was to verify the bulk viscosity bound in strongly coupled gauge theories conjectured in [31]:

$$\frac{\zeta}{\eta} \ge 2\left(\frac{1}{p} - c_s^2\right), \tag{4.13}$$

where p is the dimension of space of the plasma. It is known that (4.13) is satisfied in all explicit realizations of gauge theory/string theory correspondence [19, 31]. The bound is saturated for all Dp branes [31, 36]. It is also known that one can engineer phenomenological models motivated by gauge/string correspondence that would violate the bulk viscosity bound [37]; finally, the bound is violated in weakly coupled gauge

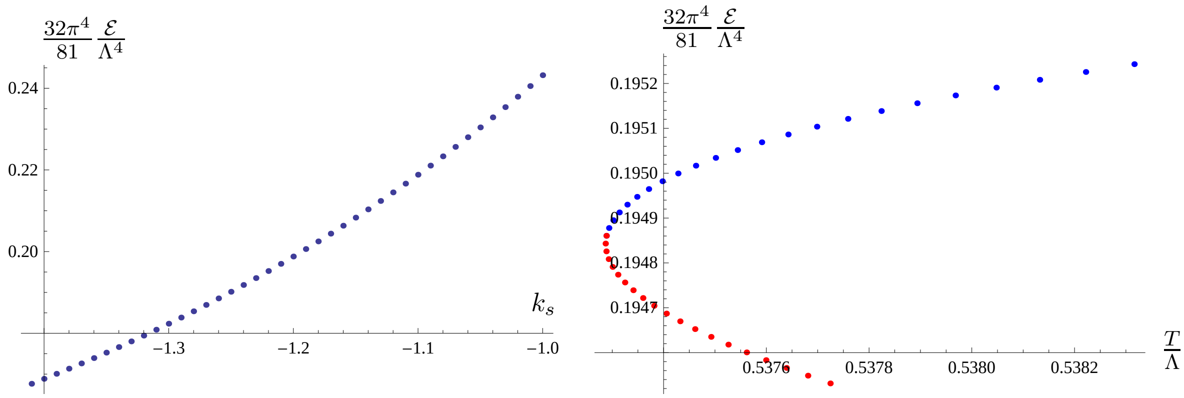

Figure 4: (color online) The energy density in the vicinity of the perturbative instability of the cascading plasma. The red dots on the right plot correspond to $k_s \le k_{unstable}$ and the blue dots correspond to $k_s \ge k_{unstable}$.

theory plasmas [38].

What makes cascading gauge theory plasma interesting in this context is that it provides an example in the framework of the gauge/string duality where the bound (4.13) is saturated to leading order in the high temperature expansion [29]. As we explained in section 3 our current computational framework is inadequate to compute bulk viscosity of the cascading plasma at low temperatures. In the high temperature expansion it is given by (see (2.14))

$$\frac{\zeta}{\eta} = \frac{8}{9}\ \frac{P^2}{K_\star} + \frac{4}{3}\left(\beta_{2,2}\ \frac{P^4}{K_\star^2} + \beta_{2,3}\ \frac{P^6}{K_\star^3} + \beta_{2,4}\ \frac{P^8}{K_\star^4}\right) + \mathcal{O}\left(\frac{P^{10}}{K_\star^5}\right)\ . \tag{4.14}$$

Using (4.6) is can be rewritten as

$$\begin{split}\frac{\zeta}{\eta} =&\delta + \left(\frac{27}{16}\beta_{1,2} + \frac{3}{8} + \frac{27}{16}\beta_{2,2}\right)\delta^2 + \left(\frac{243}{128}\beta_{1,3} + \frac{729}{128}{\beta_{1,2}}^2 + \frac{81}{64}\beta_{1,2} + \frac{9}{32}\right.\\ &+ \left.\frac{729}{128}\beta_{2,2}\beta_{1,2} + \frac{81}{64}\beta_{2,2} + \frac{243}{128}\beta_{2,3}\right)\delta^3 + \left(\frac{1215}{1024}\beta_{1,2} + \frac{32805}{2048}\beta_{1,2}\beta_{1,3} + \frac{98415}{4096}{\beta_{1,2}}^3\right.\\ &+ \frac{6561}{1024}{\beta_{1,2}}^2 + \frac{2187}{1024}\beta_{1,3} + \frac{135}{512} + \frac{2187}{1024}\beta_{1,4} + \frac{6561}{1024}\beta_{2,2}\beta_{1,3} + \frac{98415}{4096}\beta_{2,2}{\beta_{1,2}}^2\\ &+ \left.\frac{6561}{1024}\beta_{2,2}\beta_{1,2} + \frac{1215}{1024}\beta_{2,2} + \frac{2187}{1024}\beta_{2,4} + \frac{19683}{2048}\beta_{2,3}\beta_{1,2} + \frac{2187}{1024}\beta_{2,3}\right)\delta^4\\ &+ \mathcal{O}\left(\delta^5\right)\ ,\end{split} \tag{4.15}$$

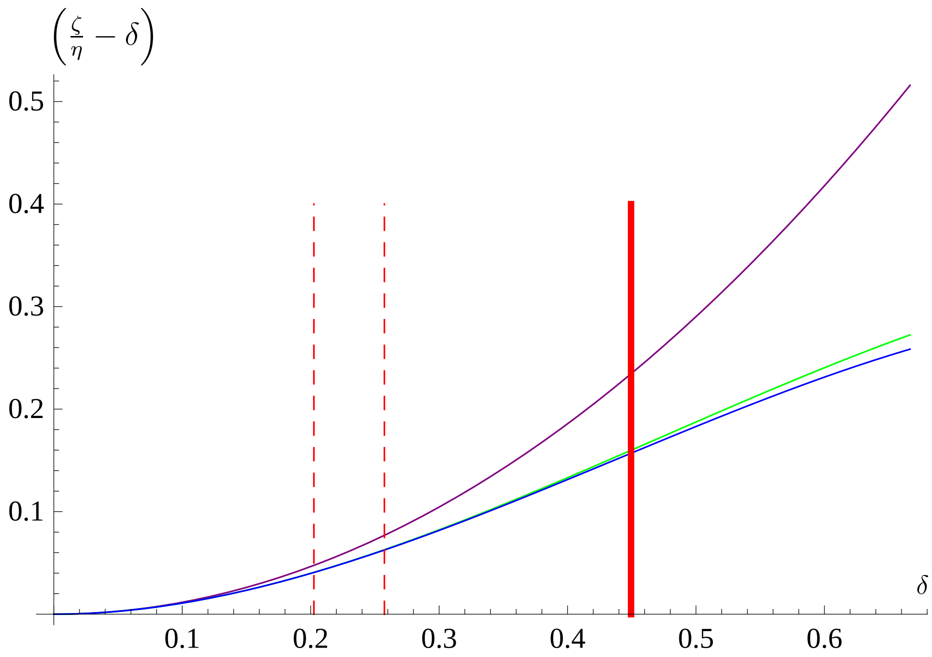

Figure 5: (color online) Successive high temperature approximations for the bulk viscosity bound for the cascading plasma. The viscosity bound implies $\frac{\zeta}{\eta} \geq \delta$. The solid vertical red line represents the temperature of the deconfinement phase transition in the cascading plasma. Vertical dashed lines indicated the expected convergence of the high temperature expansion: the left line corresponds to $T \approx 2.34 T_{critical}$ and the right line corresponds to $T \approx 1.63 T_{critical}$.

where we introduced

$$\delta \equiv 2\left(\frac{1}{3} - c_s^2\right) . \tag{4.16}$$

The results of the high temperature computations are presented in figure 5. The solid lines represent successive high temperature approximations to the bulk viscosity bound in the cascading plasma (4.15) to orders $\mathcal{O}\left(\frac{P^4}{K_\star^2}\right)$ (purple), $\mathcal{O}\left(\frac{P^6}{K_\star^3}\right)$ (green) and $\mathcal{O}\left(\frac{P^8}{K_\star^4}\right)$ (blue). Recall that the bound is exactly saturated at order $\mathcal{O}\left(\frac{P^2}{K_\star}\right)$. The vertical solid red line represents the value of $\delta = \delta_{critical}$ at the deconfinement phase transition. The dashed red lines indicate the expected convergence of the high temperature expansion of the hydrodynamic quantities deduced from the high temperature expansion of the speed of sound, see figure 1.

From figure.5 we see that at least in the high temperature expansion the bulk viscosity bound (4.13) is satisfied. Since the deconfinement phase transition in the

cascading plasma is of the first order [22], we do not expect any singular behavior in the bulk viscosity [32] in the vicinity of the transition. Notice that there is almost no difference between $\mathcal{O}\left(\frac{P^6}{K_\star^3}\right)$ (green) and $\mathcal{O}\left(\frac{P^8}{K_\star^4}\right)$ (blue) approximations to the viscosity bound all the way to the deconfinement temperature $T_{critical}$. This suggests that the high temperature expansion for the bulk viscosity might have better convergence properties than that of the speed of sound. If we take the high temperature results at the deconfinement transition seriously, we find that

$$\frac{\zeta}{\eta}\bigg|_{deconfinement} \simeq 0.6(1)\,. \tag{4.17}$$

QCD slightly above the deconfinement phase transition is nearly conformal. For c_s^2 in the range $0.27 - 0.31$, as in QCD at $T = 1.5T_{deconfinement}$ [39,40], we are well inside the expected validity range of the high temperature expansion, resulting in

$$\frac{\zeta}{\eta}\bigg|_{QGP} \approx 0.05 - 0.14 \tag{4.18}$$

for the cascading gauge theory plasma.

5 Conclusion

In this paper we presented detailed analysis of the transport properties of the deconfined chirally symmetric phase of the cascading plasma at strong coupling, using the gauge theory/string theory correspondence. We developed the high temperature expansion to order $\mathcal{Q}^4 \simeq \left(\ln \frac{T}{\Lambda}\right)^{-4}$ for the thermodynamic and the hydrodynamic properties of the theory and identified challenges in going beyond perturbative in $\mathcal{Q}$ hydrodynamics.

We computed the high temperature expansion of the bulk viscosity of the cascading plasma. We showed that the bulk viscosity bound proposed in [31] is satisfied in such plasma. We argued that results for the bulk viscosity are likely to be reliable up to the deconfinement temperature with bulk viscosity being about 60% of the shear viscosity right at the deconfinement transition. Much like in other holographic models of gauge theory/string theory duality [19] we observe a rapid drop in the bulk viscosity above the deconfinement transition.

An interesting byproduct of our hydrodynamic analysis was the discovery of the perturbative instability of the deconfined chirally symmetric phase of the cascading

plasma. Specifically, extending analysis of [22] we identified a continuous phase transition in the plasma (slightly below the critical temperature of the first order deconfinement transition) where the speed of sound squared vanishes and becomes negative. A similar phase transition was observed previously in $\mathcal{N} = 2^*$ gauge theory plasma [19,31]. Although in the former case it is difficult to speculate as to the origin of the instability, it is tempting to relate the same instability in the cascading plasma with the development of the chiral condensates responsible for the breaking of chiral symmetry. The fluctuations of such condensates are massive at high temperatures [24]. Exactly for this reason there is no high temperature regime for the deconfined cascading plasma with broken chiral symmetry — correspondingly, there can not exist a black hole solution on the warped deformed conifold with fluxes [2] at high temperatures. Of course, this does not exclude the possibility of such a black hole solution at low temperatures. The hydrodynamic stability of the symmetric phase all the way down to the deconfinement transition suggests though that the existence of the black hole in the broken phase would not modify the cosmological scenario proposed in [41]. We return to these questions in more details in future work [24].

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